Practice Problems for Final Exam (MATH 401)

Vignon S. Oussa

April 26, 2024

Use Recursive Definitions to Prove $1 + 1 = 2$

Given the recursive definitions of addition and multiplication on $\mathbb{N}$:

- $a + 0 = a$
- $a + S(b) = S(a + b)$

And for multiplication:

- $n \times 0 = 0$,
- $n \times S(m) = n + (n \times m)$.

Problem

Use these definitions to prove that $1 + 1 + 1 + 1 = 2(2)$.

Problem 2

Rational numbers as ordered pairs

Let $(a, b), (c, d) \in \mathbb{Z} \times \mathbb{Z} \setminus \{0\}$. We say that $(a, b) = (c, d)$ if and only if $ad = bc$. Show that the relation $=$ is an equivalence relation. That is, it is reflexive, symmetric, and transitive.

Show that the relation $=$ is an equivalence relation. That is, it is:

- **Reflexive:** Every element is related to itself.
- **Symmetric:** If one element is related to another, then the second is related to the first.
- **Transitive:** If one element is related to a second, and the second is related to a third, then the first is related to the third.

Cauchy Sequence Proof

Problem

Let $(a_n)_{n=1}^{\infty}$ be a sequence defined by:

$$a_n = \frac{1}{n^2 + n - 1}.$$

Prove that $(a_n)_{n=1}^{\infty}$ is a Cauchy sequence.

Proof by Mathematical Induction

Problem

Prove the following statement by appealing to mathematical induction:

$$\sum_{k=1}^{n} (-2)^k 2^{2k} = \frac{8(-8)^n - 8}{9}$$

for every natural number n.

Convergence of a Bounded Sequence

Problem

Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ such that for all $n \in \mathbb{N}$,

$$1 \leq a_n \leq 2.$$

Prove that if $(a_n)_{n\in\mathbb{N}}$ is convergent, then $\lim_{n\to\infty} a_n \leq 2$.

Convergence of a Sequence

Problem

Prove formally that the sequence with terms

$$a_n = 1 + (-1)^n \cos\left(\frac{2\pi n}{3}\right) \cdot \frac{1}{n}$$

converges to 1 as n approaches infinity.

Divergence and Boundedness of a Sequence

Problem

Prove the following properties of the sequence $(a_n)_{n=0}^{\infty}$*, where* $a_n = \cos\left(\frac{2\pi n}{3}\right) - \sin\left(\frac{2\pi n}{4}\right)$ *for* $n = 0, 1, 2, \ldots$*:*

- *The sequence is not monotonic.*
- *The sequence is divergent.*
- *The sequence is bounded.*
- *Discuss whether this is a properly divergent sequence and clearly justify your answer.*

Proving a Limit Using the ε-N Criterion

Problem

Prove formally by using the ε-N criterion that

$$\lim_{n\to\infty}\left(\frac{n^3\cos(n)}{n^5-n^2+4}\right)=0$$

Problem

Let $(a_n)_{n=1}^{\infty}$ *be a sequence recursively defined by:*

$$a_{n+1} = \frac{a_n}{a_n + 1}, \quad a_1 = 2.$$

1. *Compute the first five terms of the sequence.*
2. *Prove that the sequence is bounded.*
3. *Prove that the sequence is monotone.*
4. *Prove that the sequence is convergent.*
5. *Compute its limit.*

Limit Proof Using the ε-δ Criterion

Problem

Let $f(x) = x^3 - x$. *Prove formally, using the* ε-δ *criterion, that*

$$\lim_{x \to 1} f(x) = 0.$$

Epsilon-Delta Limit Proof

Problem

Let $f(x) = \frac{1}{x^2+1}$*. Prove formally, using the* ε*-*δ *criterion, that*

$$\lim_{x \to 1} f(x) = \frac{1}{2}.$$

Continuity Analysis

Problem

Define the function f as follows:

$$f(x) = \begin{cases} x^3 \sin\left(\cos\left(\sin\left(\frac{1}{x^{100}}\right)\right)\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Analyze the continuity of f at $x = 0$.