Appendix A (notes)

- A **statement** is a declarative sentence which can be assigned a true or false value but not both.
  - The sky is blue can be either a true or false. Therefore it is a statement
  - Dogs are cute is a matter of opinion. As such, this sentence is not a statement.
- Two statements P and Q are **logically equivalent** if they hold the same truth value.
- If P is a statement then its **negation** is denoted not P. The negation of a statement and the given statement always have oppositive truth value.
- Given two statements P,Q, their **conjunction** is the statement P and Q which is true when both statements are true and false otherwise.
- Given two statements P,Q, their **disjunction** is the statement P or Q which is true if and only if at least one of the given statement is true.
  - For example, $\sqrt{2} > \sqrt{3}$ or $\sqrt{2} < \sqrt{3}$ is a true statement
- The **implication (or conditional)** of two statements P,Q is the statement P implies Q or if P then Q or $P \Rightarrow Q$ having the following truth table

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

  It is worth noting that the negation of the statement $P \Rightarrow Q$ is given by the statement P and not Q

  - For example, if a function is differentiable then the function must be continuous is an implication. Is this a true or false statement?
  - The first statement P is called the **hypothesis** and the second statement Q is called the **conclusion** of the implication or conditional statement.
  - The **contrapositive** of $P \Rightarrow Q$ is not $Q \Rightarrow$ not P. It is also a known fact that an implication and its contrapositive both have the same truth value
  - The **converse** of a statement $P \Rightarrow Q$ is the statement $Q \Rightarrow P$ and it is important not to confuse the converse with the contrapositive

- Given statements P, Q, the **biconditional** statement P if and only Q denoted $P \Leftrightarrow Q$ is the statement obtained by taking the conjunction of the implication $P \Rightarrow Q$ and its converse. The statement $P \Leftrightarrow Q$ is true precisely when both P and Q have the same truth value.

- Consider the following statement: For every $x \in \mathbb{R}, x^2 \geq 0$. For every known as the **universal quantifier** is often denoted $\forall$. On the other hand, the symbol $\exists$ is the **existential quantifie**r which means means there exists.

  - Example: For every $\epsilon > 0$, the interval $(-\epsilon, \epsilon)$ contains a rational number
  - Example: there exists a positive real numer ϵ such that $\sqrt{2}$ belongs to the open interval $(-\epsilon, \epsilon)$.

- A **direct proof** is a proof technique which can be exploited to establish that an implication of the type If P then Q is a true statement. In this approach, we always start by assuming that the hypothesis holds, and use this assumption to derive that the conclusion of the implication is true.

  - **Claim** If x is an odd integer then x^2 is odd as well.
    * **Proof (direct proof)** Suppose that x is an odd integer. Then there exists an integer k such that $x = 2k + 1$. Next squaring each side of the equation $x = 2k + 1$ gives
$$x^2 = 4k^2 + 4k + 1.$$
      Factoring 2 of out of first two terms yields
$$x^2 = 2\left(2k^2 + 2k\right) + 1.$$
      Therefore x^2 is an odd integer.

- A **proof by contrapositive** is a direct proof of the contrapositive of the given statement. This proof strategy hinges on the fact that an implication and its contrapositive have the same truth value

  - **Claim** Let $x \geq 0$. If for every $\epsilon > 0$ we have $0 \leq x < \epsilon$ then $x = 0$
  - **Proof** (by contrapositive) Suppose that $x > 0$. Next, set $\epsilon = \frac{x}{2}$. Then clearly
$$\epsilon = \frac{x}{2} < x.$$
    Thus, there exists a positive real number $\epsilon > 0$ such that $\epsilon < x$. Thus our hypothesis is false.

- A **proof by contradiction** of a statement of the type $P \Rightarrow Q$ hinges upon the fact that the negation of $P \Rightarrow Q$ is false. Precisely, it requires showing that the statement P and not Q cannot be true.

- **Claim** Let $x > 0$. If $x > 0$ then $\frac{1}{x} > 0$
- **Proof** (by contradiction) Suppose that there exists a positive number x whose inverse is not positive. In other words $\frac{1}{x} \leq 0$. Since $x > 0$ it follows that

$$\left(\frac{1}{x}\right) x \leq (0)\, x$$

and consequently,

$$1 \leq 0$$

violating the ordering structure of the real numbers.