

Solutions for review homework

1. **(Direct Proof)** Assume that x and y are odd integers. Then there exist integers u and v such that

$$x = 2u + 1 \text{ and } y = 2v + 1.$$

Next,

$$\begin{aligned} xy &= (2u + 1)(2v + 1) \\ &= 4uv + 2u + 2v + 1 \\ &= 2(2uv + u + v) + 1. \end{aligned}$$

Put

$$w = 2uv + u + v.$$

Then

$$xy = 2w + 1.$$

Thus, xy is an odd integer.

2. **(Direct Proof)** Let a and b be positive real numbers. Assume that

$$a < b.$$

Then

$$b^2 - a^2 = (b - a)(b + a).$$

Now, since $a < b$ then

$$a - b < 0$$

and it follows that

$$b - a > 0.$$

Additionally, since a and b are positive integers then clearly,

$$a + b > 0 \text{ (the sum of two positive numbers is positive).}$$

Next, since the product of two numbers of the same sign is positive then

$$b^2 - a^2 = (b - a)(b + a) > 0.$$

This completes the proof.

3. **(Proof by contradiction).** Assume that there exists an integer m such that m^2 is even and m is odd. Then

$$m = 2n + 1$$

for some integer n . Now, squaring each side of the equation above gives

$$m^2 = (2n + 1)^2.$$

Next, foiling $(2n + 1)^2$, we obtain

$$\begin{aligned} m^2 &= (2n + 1)^2 \\ &= 4n^2 + 4n + 1 \\ &= 2(2n^2 + 2n) + 1 \end{aligned}$$

and it follows that m^2 is odd, and this contradicts our assumption. Thus, it is not possible to find an odd number whose square is even. This completes the proof.

4. **(Proof by contradiction)** Let us suppose that $\sqrt{2}$ is a rational number. That is, there exist nonzero integers n, m such that

$$\sqrt{2} = \frac{n}{m}.$$

We shall furthermore assume that the fraction $\frac{n}{m}$ is given in its reduced form (n and m have no common divisor). Next, squaring each side of the equation above gives:

$$2 = \frac{n^2}{m^2}.$$

It follows that

$$n^2 = 2m^2. \tag{1}$$

Thus n^2 is an even integer. Now, since n^2 is even then n is also even (see Problem 1). So, there exists an integer k such that

$$n = 2k$$

and this implies that

$$n^2 = 4k^2.$$

Next, coming back to (1) we obtain

$$4k^2 = n^2 = 2m^2.$$

So,

$$m^2 = 2k^2$$

and this implies that m is also even. Thus, 2 divides both n and m . However this is impossible since the fraction

$$\frac{n}{m}$$

is in its reduced form (the integers n, m have no common factor).

5. There are **two** statements to prove here. First, we need to show that if m and n have the same parity then $m^2 + n^2$ is even. Secondly, we must also show that if $m^2 + n^2$ is even then m and n have the same parity. For the first statement, let m and n be integers. Assume that m and n have the same parity. There are two cases to consider. First let us assume that m, n are both even. That is,

$$m = 2k \text{ and } n = 2j$$

for some integers k, j . Then

$$\begin{aligned} m^2 + n^2 &= (2k)^2 + (2j)^2 \\ &= 4j^2 + 4k^2 \text{ (this is an even number).} \end{aligned}$$

Secondly, assume that m, n are both odd. That is

$$m = 2k + 1 \text{ and } n = 2j + 1$$

for some integers k, j . Then

$$\begin{aligned} m^2 + n^2 &= (2k + 1)^2 + (2j + 1)^2 \\ &= 4j^2 + 4j + 4k^2 + 4k + 2 \\ &= 2(2j^2 + 2j + 2k^2 + 2k + 1) \end{aligned}$$

which is also clearly even. For the second part of the proof, we want to show that if $m^2 + n^2$ is even then m and n have the same parity. By way of contradiction, assume that $m^2 + n^2$ is even and m and n do not have the same parity. Then, it must be the case that

$$m^2 + n^2 = (2k + 1)^2 + (2j)^2$$

for some integers k and j . Observe that

$$\begin{aligned} m^2 + n^2 &= (2k + 1)^2 + (2j)^2 \\ &= 4j^2 + 4k^2 + 4k + 1 \\ &= 2(2j^2 + 2k^2 + 2k) + 1. \end{aligned}$$

Thus, $m^2 + n^2$ is odd and this contradicts our assumption.