

Section 1.2 (solutions)

Claim 1 For all natural numbers n

$$1^2 + 3^2 + \cdots + (2n - 1)^2 = \frac{4n^3 - n}{3}.$$

Proof. For the basis of induction, we verify that whenever $n = 1$

$$\frac{4(1)^3 - 1}{3} = 1$$

and

$$(2(1) - 1)^2 = 1$$

Therefore the basis of induction holds. For the inductive step, we need to prove that if

$$1^2 + 3^2 + \cdots + (2k - 1)^2 = \frac{4k^3 - k}{3}$$

then

$$1^2 + 3^2 + \cdots + (2k - 1)^2 + (2(k + 1) - 1)^2 = \frac{4(k + 1)^3 - (k + 1)}{3}.$$

To this end, let us suppose that $1^2 + 3^2 + \cdots + (2k - 1)^2 = \frac{4k^3 - k}{3}$ holds. Then

$$\begin{aligned} & 1^2 + 3^2 + \cdots + (2k - 1)^2 + (2(k + 1) - 1)^2 \\ \text{(by the inductive hypothesis)} &= \underbrace{1^2 + 3^2 + \cdots + (2k - 1)^2}_{\frac{4k^3 - k}{3}} + (2(k + 1) - 1)^2 \\ &= \frac{4k^3 - k}{3} + (2(k + 1) - 1)^2 \\ \text{(common denominator)} &= \frac{4k^3 - k + 3(2(k + 1) - 1)^2}{3} \\ \text{(expanding the numerator)} &= \frac{4k^3 + 12k^2 + 11k + 3}{3} = (*) \end{aligned}$$

Next, note that

$$4(k+1)^3 - (k+1) = 4k^3 + 12k^2 + 11k + 3$$

Consequently,

$$(*) = \frac{4(k+1)^3 - (k+1)}{3}$$

Thus, we conclude that for every natural number n

$$1^2 + 3^2 + \dots + (2n-1)^2 = \frac{4n^3 - n}{3}.$$

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Claim 2 For every natural number n , $n^3 + 5n$ is divisible by 6.

Proof. For the basis of induction, we verify that if $n = 1$ then $n^3 + 5n = 6$ is clearly divisible by 6. To establish the inductive step we need to prove that the following implication holds true. If $k^3 + 5k$ is divisible by 6 then $(k+1)^3 + 5(k+1)$ is divisible by 6. To this end, suppose that for some natural number k , $k^3 + 5k$ is divisible by 6. Then there is a natural number ℓ such that

$$k^3 + 5k = 6\ell.$$

As such,

$$\begin{aligned}(k+1)^3 + 5(k+1) &= k^3 + 3k^2 + 8k + 6 \\ &= 6\ell - 5k + 3k^2 + 8k + 6 \\ &= 3k^2 + 3k + 6\ell + 6 \\ &= 3k(k+1) + 6\ell + 6 = (*).\end{aligned}$$

Next, note that since the quantity $k(k+1)$ is a product of an even number and an odd one, then it must be even as well. In other words, there is a natural number m such that

$$k(k+1) = 2m$$

Thus,

$$(*) = 3(2m) + 6\ell + 6 = 6m + 6\ell + 6 = 6(m + \ell + 1)$$

and we conclude that $(k+1)^3 + 5(k+1)$ is a multiple of six. ■

Claim 3 For every natural number n , $5^{2n} - 1$ is divisible by 8.

Proof. For the basis of induction, we verify that if $n = 1$ then

$$5^{2n} - 1 = 5^2 - 1 = 24$$

is a multiple of eight. Next, for the inductive step, we need to prove that if $5^{2k} - 1$ is divisible by 8 for some natural number k then $5^{2(k+1)} - 1$ is divisible by 8. To this end, suppose that

$$5^{2k} - 1 = 8j \text{ for some integer } j.$$

Then

$$\begin{aligned} 5^{2(k+1)} - 1 &= 25 \times 5^{2k} - 1 \\ &= 25(8j + 1) - 1 \\ &= 200j + 24 \\ &= 8(25j + 3) \end{aligned}$$

Thus, $5^{2(k+1)} - 1$ is divisible by 8. ■

Claim 4 For every natural number n , $n < 2^n$.

Proof. The basis of induction clearly holds since $1 < 2$. For the inductive hypothesis, we need to prove that if $k < 2^k$ for some natural number k then $k + 1 < 2^{k+1}$. Suppose that $k < 2^k$. Adding one to each side of the inequality gives

$$k + 1 < 2^k + 1$$

Since k is a natural number,

$$2^k + 1 < 2^k + 2^k = 2(2^k) = 2^{k+1}$$

Thus,

$$k + 1 < 2^k + 1 < 2^{k+1}$$

and we conclude that for every natural number n , $n < 2^n$. ■

Claim 5 $2^n < n!$ for all natural numbers $n \geq 4$. (Recall that $n! = n(n-1) \cdots (2)(1)$).

Proof. For the basis of induction, it suffices to verify that

$$2^4 = 16 < 4! = 24$$

To establish the inductive step we need to show that the following implication holds

$$2^k < k! \Rightarrow 2^{k+1} < (k+1)!$$

for $k \geq 4$. To this end, suppose that $2^k < k!$ Multiplying each side of the inequality by two gives

$$2^{k+1} < 2(k!)$$

Since $2 < k+1$ it follows that

$$2(k!) < (k!)(k+1) = (k+1)!$$

Thus, we conclude that

$$2^{k+1} < (k+1)!. \quad \blacksquare$$

Claim 6 For every natural number $n > 1$

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} > \sqrt{n}.$$

Proof. For the basis of induction, we verify that for $n = 2$

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} = \frac{\sqrt{2} + 1}{\sqrt{2}} > \sqrt{2}.$$

Next, suppose that for some natural number $k > 1$

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{k}} > \sqrt{k}.$$

Then

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} > \sqrt{k} + \frac{1}{\sqrt{k+1}}$$

Next we note that since $\sqrt{k(k+1)} - k$ is a positive quantity

$$\left(\sqrt{k} + \frac{1}{\sqrt{k+1}} \right) - \sqrt{k+1} = \frac{\sqrt{k(k+1)} - k}{\sqrt{k+1}} > 0$$

Consequently,

$$\sqrt{k} + \frac{1}{\sqrt{k+1}} > \sqrt{k+1}$$

and it follows that

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} > \sqrt{k+1}.$$

We conclude that fevery natural number $n > 1$

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} > \sqrt{n}.$$

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