Section 3.4 Sequences and Series

An example of an unbounded sequence with a convergent subsequence.

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence with terms $x_n = n + (-1)^n \cdot n$.

Then $\forall k \in \mathbb{N}$, $x_{2k} = 2k + (-1)^{2k} \cdot (2k) = 2k + 2k = 4k$.

$\forall k \in \mathbb{N}$, $x_{2k+1} = (2k+1) + (-1)^{2k+1}(2k+1) = (2k+1) - (2k+1) = 0$.

Given any $M > 0$, appealing to the Archimedean Property, there is a $k \in \mathbb{N}$ such that $k > M/4$. Thus $4k > M$ and $x_{2k} > M$. Thus the sequence

$(n + (-1)^n n)_{n\in\mathbb{N}}$ is unbounded.

On the other hand, given $k \in \mathbb{N}$, $x_{2k+1} = 0$. Thus the subsequence $(x_{2k+1})_{k\in\mathbb{N}}$ is a convergent sequence (indeed, it is a constant sequence!)

(#3) Let $(f_n)_{n\in\mathbb{N}}$ be a sequence given by $f_1 = 1$, $f_2 = 2$, $f_{n+1} = f_{n-1} + f_n$

Next, let $x_n = \dfrac{f_{n+1}}{f_n}$. Suppose that $\lim\limits_{n\to\infty} x_n = L$.

Then $\lim\limits_{n\to\infty} \dfrac{f_{n+1}}{f_n} = L$, and $\lim\limits_{n\to\infty} \dfrac{f_{n-1} + f_n}{f_n}$

$$= \lim_{n\to\infty} \frac{f_{n-1}}{f_n} + \frac{f_n}{f_n} = \lim_{n\to\infty} \frac{f_{n-1}}{f_n} + 1$$

$$= \lim_{n\to\infty} \frac{1}{\frac{f_n}{f_{n-1}}} + 1 = \frac{1}{\lim\limits_{n\to\infty} \frac{f_n}{f_{n-1}}} + 1 = \frac{1}{L} + 1 = L.$$

Solving the equation $\frac{1}{L} + 1 = L$

$$\frac{1}{L} + 1 = L \iff 1 + L = L^2$$
$$\iff L^2 - L - 1 = 0$$
$$\iff L = \frac{1 + \sqrt{1+4}}{2} \text{ or } L = \frac{1 - \sqrt{1+4}}{2}$$
$$\iff L = \frac{1+\sqrt{5}}{2} \text{ or } L = \frac{1-\sqrt{5}}{2}$$

Since $\frac{1+\sqrt{5}}{2} > 0$ and $\frac{1-\sqrt{5}}{2} < 0$

$$\boxed{\lim_{n\to\infty} \frac{f_{n+1}}{f_n} = \frac{1+\sqrt{5}}{2}}$$

(4a) Let $x_n = 1-(-1)^n + \frac{1}{n}$.

Then $x_{2K} = 1-1+\frac{1}{2K} = \frac{1}{2K}$ and $x_{2K+1} = 1+1+\frac{1}{2K+1} = 2+\frac{1}{2K+1}$

$$= \frac{4K+3}{2K+1}.$$

Moreover $\lim_{K\to\infty} x_{2K} = \lim_{K\to\infty} \frac{1}{2K} = 0$,

$$\lim_{K\to\infty} x_{2K+1} = \lim_{K\to\infty} \frac{4K+3}{2K+1} = 2.$$

$(x_n)_{n\in\mathbb{N}}$ has 2 subsequences converging to 2 different limits $2 \neq 0$. Thus $(x_n)_{n\in\mathbb{N}}$ is a diverging sequence.

(4b) Let $(y_n)_{n\in\mathbb{N}}$ be a sequence with terms $y_n = \sin\frac{n\pi}{4}$.

Then $y_{4K} = \sin\left(\frac{4K\pi}{4}\right) = \sin(K\pi) = 0 \quad \forall K\in\mathbb{N}$.

Next, $y_{8K+2} = \sin\left(\frac{(8K+2)\pi}{4}\right) = \sin\left(\frac{8K\pi}{4} + \frac{2\pi}{4}\right)$

$= \sin\left(2K\pi + \frac{\pi}{2}\right) = \sin(\pi/2)$ since the sine function is periodic with period 2π. Therefore, $\forall K\in\mathbb{N}$, $y_{8K+2} = \sin\frac{\pi}{2} = 1$. In light of the observations above, $(y_n)_{n\in\mathbb{N}}$ has 2 subsequences converging to 2 different limits. Indeed $\lim_{K\to\infty} y_{4K} = 0$ and

$\lim_{K\to\infty} y_{8K+2} = 1 \neq 0$!

(8a) $\lim\limits_{n\to\infty}(3n)^{\frac{1}{2n}} = ?$

$(3n)^{\frac{1}{2n}} = e^{\frac{1}{2n}\ln(3n)}$. Now, $\lim\limits_{n\to\infty} \dfrac{\ln(3n)}{2n} = \lim\limits_{n\to\infty} \dfrac{\frac{3}{3n}}{2}$

$= \lim\limits_{n\to\infty} \dfrac{1}{2n} = 0$

Consequently, $\lim\limits_{n\to\infty}(3n)^{\frac{1}{2n}} = \lim\limits_{n\to\infty} e^{\frac{1}{2n}\ln(3n)}$

$= \boxed{e^0 = 1}$

(8b) $\left(1+\dfrac{1}{2n}\right)^{3n} = e^{3n\ln\left(1+\frac{1}{2n}\right)}$

Next, $\lim\limits_{n\to\infty} 3n\ln\left(1+\dfrac{1}{2n}\right) = \lim\limits_{n\to\infty} \dfrac{\ln\left(1+\frac{1}{2n}\right)}{(3n)^{-1}}$

$= \lim\limits_{n\to\infty} \dfrac{\ln\left(\frac{2n+1}{2n}\right)}{(3n)^{-1}} = \lim\limits_{n\to\infty} \dfrac{\ln(2n+1) - \ln(2n)}{(3n)^{-1}}$

$= \lim\limits_{n\to\infty} \dfrac{\frac{2}{2n+1} - \frac{2}{2n}}{-(3n)^{-2}(3)} = \lim\limits_{n\to\infty} \dfrac{\frac{4n-4n-2}{(2n+1)(2n)}}{-\frac{3}{9n^2}}$

$= \lim\limits_{n\to\infty} \dfrac{2}{(2n+1)(2n)} \cdot \dfrac{9n^2}{3} = \lim\limits_{n\to\infty} \dfrac{6n^2}{4n^2+2n} = 3/2.$

thus $\lim\limits_{n\to\infty}\left(1+\frac{1}{2n}\right)^{3n} = e^{3/2}.$

Claim Suppose every subsequence of $(x_n)_{n\in\mathbb{N}}$ has a subsequence that converges to 0. Then $\lim_{n\to\infty} x_n = 0$.

Proof By contrapositive, suppose that $(x_n)_{n\in\mathbb{N}}$ does not converge to 0. Then there is a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ that does not converge to 0. Thus, there exist $\varepsilon>0$ such that for all $N\in\mathbb{N}$, there exists $n_k > N$ so that $|x_{n_k}| > \varepsilon$. Now by recursion, choose $n_1 > 1$ so that $|x_{n_1}| > \varepsilon$. then select $x_{n_{k+1}}$ so that $n_{k+1} > n_k$ so that $|x_{n_{k+1}}| > \varepsilon$. Iterating this process gives a sequence $(x_{n_1}, x_{n_2}, x_{n_3}, \dots, x_{n_k}, x_{n_{k+1}}, \dots)$ which is not converging to 0. ∎

(ii) Suppose $x_n \geq 0$ $\forall n\in\mathbb{N}$ and
$\lim_{n\to\infty} (-1)^n x_n = L$ for some $L\in\mathbb{R}$.

Let $\varepsilon > 0$. By assumption, there exists $N\in\mathbb{N}$ depending on ε such that if $n > N$ then $|(-1)^n x_n - L| < \varepsilon$.

Given $n\in\mathbb{N}$, $\big||(-1)^n x_n| - |L|\big| \leq |(-1)^n x_n - L|$ and,
$\big||(-1)^n x_n| - |L|\big| = \big||x_n| - |L|\big| = |x_n - |L||$ since $x_n \geq 0$ $\forall n\in\mathbb{N}$

Fixing N depending on ε so that if $n > N$ then $|(-1)^n x_n - L| < \varepsilon$, it follows that if we assume that $n > N$,
$\big||(-1)^n x_n| - |L|\big| = |x_n - |L|| \leq |(-1)^n x_n - L| < \varepsilon$. Thus $\lim_{n\to\infty} x_n = |L|$