Section 3.3 Monotone Sequences

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- Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. We say that $(x_n)_{n\in\mathbb{N}}$ is **increasing** if it satisfies the inequalities

$$x_1 \leq x_2 \leq \cdots \leq x_n \leq x_{n+1} \leq \cdots$$

- Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. We say that $(x_n)_{n\in\mathbb{N}}$ is **decreasing** if it satisfies the inequalities

$$x_1 \geq x_2 \geq \cdots \geq x_n \geq x_{n+1} \geq \cdots$$

- We say that a sequence is **monotone** or **monotonic** if it is either increasing or decreasing

- A sequence (s_n) of real numbers is called a **nondecreasing** sequence if

$$s_n \leq s_{n+1} \text{ for all } n$$

- A sequence (s_n) of real numbers is called a **nonincreasing** sequence if

$$s_n \geq s_{n+1} \text{ for all } n$$

- A sequence that is nondecreasing or nonincreasing will be called a **monotone sequence** or a **monotonic sequence.**

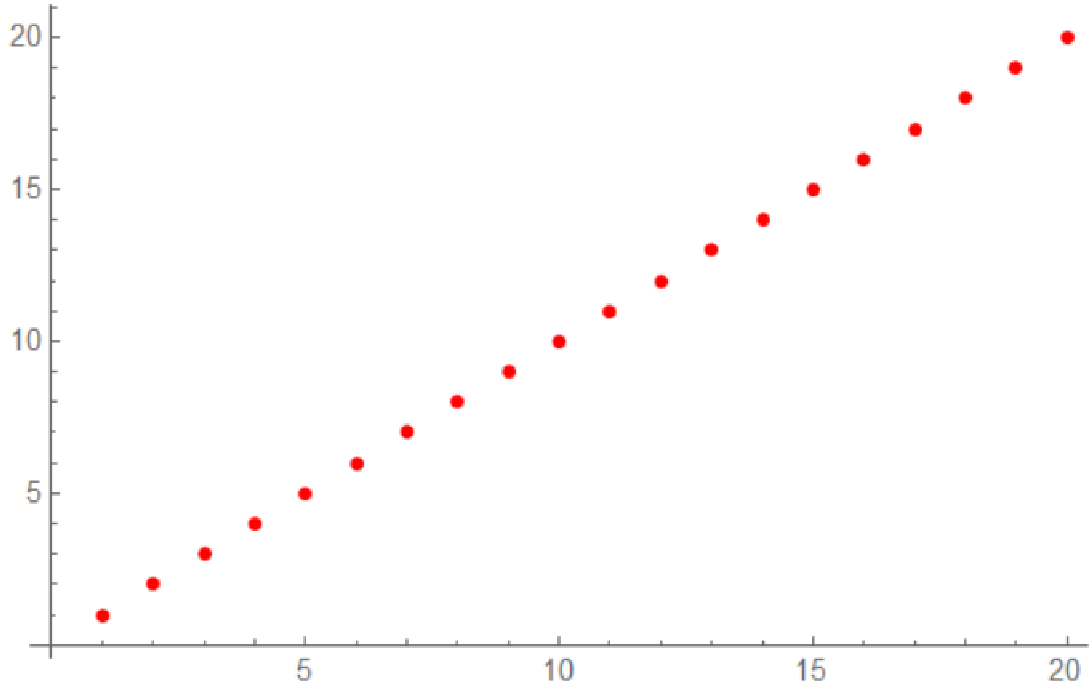
Example 1 *Let* $(x_n)_{n\in\mathbb{N}}$ *be a sequence such that*

$$x_n = n$$

Note that for every $k \in \mathbb{N}$

$$x_{k+1} = k + 1 > k = x_k.$$

Thus, the sequence $(n)_{n\in\mathbb{N}}$ *is an increasing sequence.*



Example 2 *Let* $(x_n)_{n\in\mathbb{N}}$ *be a sequence such that*

$$x_n = 3^n$$

Note that for every $k \in \mathbb{N}$

$$x_k = 3^k \text{ and } x_{k+1} = 3^{k+1}$$

Thus, the sequence $(3^n)_{n\in\mathbb{N}}$ *is an increasing sequence.*

Example 3 *Let* $(x_n)_{n\in\mathbb{N}}$ *be a sequence such that*

$$x_n = \frac{1}{n}$$

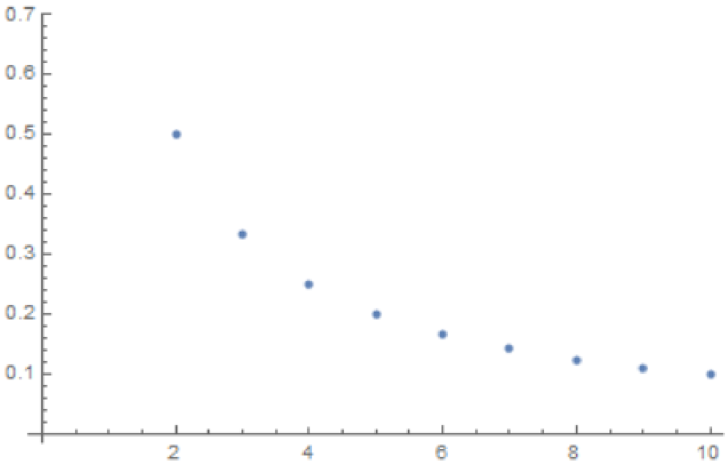
Note that for every $k \in \mathbb{N}$

$$x_k = \frac{1}{k} \text{ and } x_{k+1} = \frac{1}{k+1}$$

and

$$\frac{1}{k+1} \leq \frac{1}{k}$$

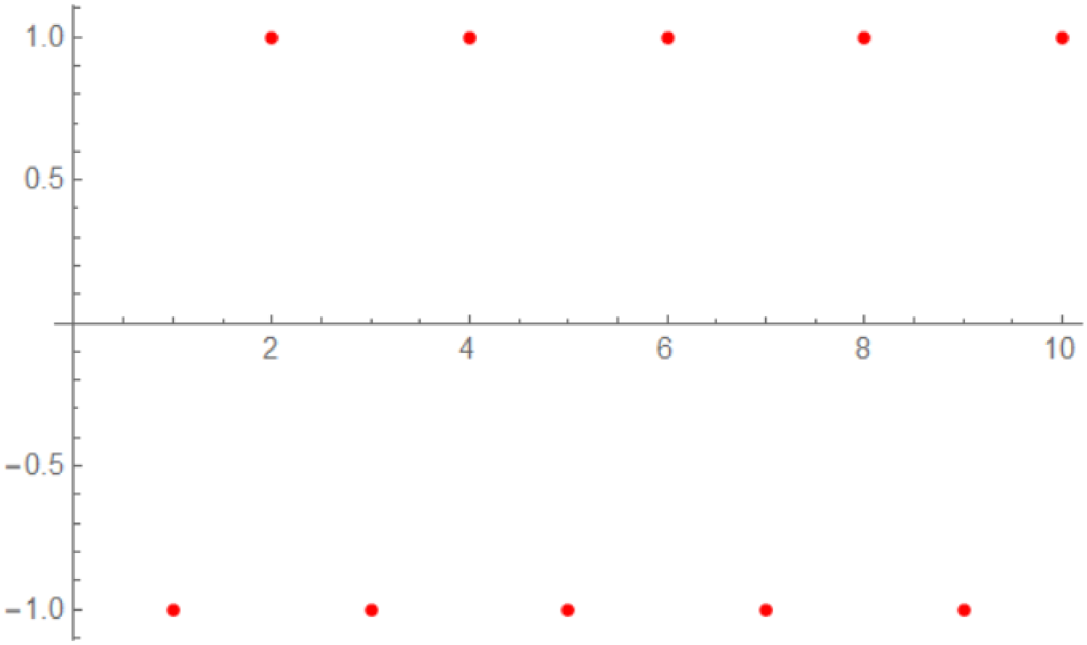
Thus, the sequence $\left(\frac{1}{n}\right)_{n\in\mathbb{N}}$ *is a decreasing sequence.*



Example 4 *The sequence*

$$((-1)^n)_{n\in\mathbb{N}}$$

is not monotonic. Can you prove this statement?



Example 5 *The sequence*

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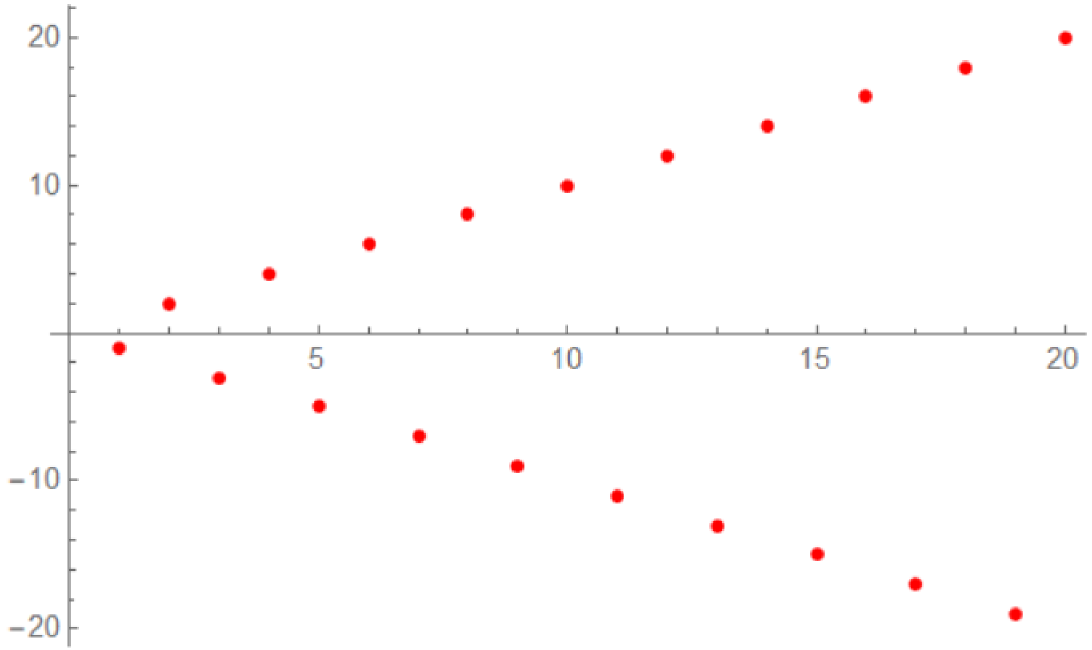

Theorem 6 *(Monotone Convergence Theorem) Any monotone and bounded sequence is convergent*

1. *Any increasing and bounded sequence converges to its supremum.*
2. *Any decreasing and bounded sequence converges to its infimum.*

Proof. Suppose that $(x_n)_{n\in\mathbb{N}}$ is a bounded, increasing sequence. Then there is a positive real number $M > 0$ such that

$$|x_n| \leq M \text{ for all } n \in \mathbb{N}.$$

Now by the Completeness Property, the supremum of the set

$$A = \{x_n : n \in \mathbb{N}\}$$

exits as a real number. Let

$$x^* = \sup(A)$$

We want to show that

$$\lim x_n = x^*.$$

In other words, we want to prove that for any $\epsilon > 0$, there is a natural number $N \in \mathbb{N}$ such that if $n > N$ then

$$|x_n - x^*| < \epsilon.$$

Now, since $x^* - \epsilon < x^*$, it is clear that $x^* - \epsilon$ is not an upperbound for A. As such, there is $x_N \in A$ such that

$$x_N > x^* - \epsilon$$

Now, since $(x_n)_{n\in\mathbb{N}}$ is increasing then if $n > N$ then

$$x_n \geq x_N > x^* - \epsilon$$

and consequently,

$$x^* - \epsilon < x_N \leq x_n \leq x^* < x^* + \epsilon.$$

Thus,

$$x^* - \epsilon < x_n < x^* + \epsilon$$

and

$$-\epsilon < x_n - x^* < \epsilon.$$

This implies that for $n > N$ we have

$$|x_n - x^*| < \epsilon.$$

We conclude that

$$\lim x_n = x^*$$

For the second part, if $(y_n)_{n\in\mathbb{N}}$ is a bounded decreasing sequence, then $(-y_n)_{n\in\mathbb{N}}$ is a bounded increasing sequence and from the first part, we know that

$$\begin{aligned}\lim_{n\to\infty} -y_n &= \sup\{-y_n : n \in \mathbb{N}\}\\ &= -\inf\{y_n : n \in \mathbb{N}\}\end{aligned}$$

Thus,

$$\begin{aligned}-\lim_{n\to\infty} -y_n &= \inf\{y_n : n \in \mathbb{N}\}\\ &= \lim_{n\to\infty} y_n\end{aligned}$$

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Example 7 *We want to prove that*

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$$

First, let us prove that this sequence is decreasing. Given any natural number n

$$\sqrt{n} \leq \sqrt{n+1}$$

Next,

$$\frac{1}{\sqrt{n}} \geq \frac{1}{\sqrt{n+1}}.$$

Thus,

$$\left(\frac{1}{\sqrt{n}}\right)_{n\in\mathbb{N}}$$

is decreasing. Secondly, we need to prove that $\left(\frac{1}{\sqrt{n}}\right)_{n\in\mathbb{N}}$ *is a bounded sequence. Given any natural number* $n \in \mathbb{N}$

$$n \geq 1 \Rightarrow \sqrt{n} \geq 1 \Rightarrow \frac{1}{\sqrt{n}} \leq 1$$

As such, $\left(\frac{1}{\sqrt{n}}\right)_{n\in\mathbb{N}}$ *is decreasing and bounded. From the previous result, we obtain*

$$\lim_{n\to\to\infty} \frac{1}{\sqrt{n}} = \inf\left\{\frac{1}{\sqrt{n}} : n \in \mathbb{N}\right\} = 0.$$

Example 8 *Let* $(y_n)_n$ *be a sequence given by*

$$y_1 = 1 \text{ and } y_{n+1} = \frac{1}{4}(2y_n + 3)$$

We want to show that

$$\lim_{n\to\infty} y_n = \frac{3}{2}.$$

First,

n	y_n
1	1
2	1.25
3	1.375
4	1.4375
5	1.46875
6	1.48438
7	1.49219

The table above suggests that

$$y_n \leq \frac{3}{2} \text{ for every natural number } n.$$

Proving this claim by induction, the basis of induction holds by definition. Next, for the inductive step, suppose that

$$y_k \leq \frac{3}{2} \text{ for some } k \in \mathbb{N}, k \geq 1.$$

Now,

$$\begin{aligned} y_{k+1} &= \frac{1}{4}(2y_k + 3) \\ &\leq \frac{1}{4}\left(2\left(\frac{3}{2}\right) + 3\right) \text{ by inductive hypothesis} \\ &\leq \frac{3}{2} \end{aligned}$$

and this is the desired result. Thus, $(y_n)_n$ *is a bounded sequence.*

Next we shall prove by induction that

$$y_n < y_{n+1} \textit{ for all } n \in \mathbb{N}$$

In other words, this recursively defined sequence is increasing. For the basis of induction if $n = 1$, we verify that

$$y_1 = 1 < y_2 = \frac{5}{4}.$$

Now suppose that

$$y_k < y_{k+1} \textit{ for some } k \in \mathbb{N}, k \geq 1$$

Now,

$$\begin{aligned} y_{k+2} &= \frac{1}{4}\left(2y_{k+1} + 3\right) \\ &> \frac{1}{4}\left(2y_k + 3\right) \ \textit{ By the inductive hypothesis} \\ &= y_{k+1} \end{aligned}$$

and this gives us the desired result. Thus, the given sequence is bounded and also increasing. It follows from the Monotone Convergence Theorem that $(y_n)_n$ is convergent to some real numbers. Now, let

$$y = \lim_{n\to\infty} y_n.$$

Then

$$\begin{aligned} \lim_{n\to\infty} y_{n+1} &= \lim_{n\to\infty} \frac{1}{4}\left(2y_n + 3\right) = y \\ &= \frac{1}{4}\left(2y + 3\right). \end{aligned}$$

Thus,

$$\frac{1}{4}\left(2y + 3\right) = y.$$

Solving the above equation for y gives $y = \frac{3}{2}$. We conclude that

$$\lim_{n\to\infty} y_n = \frac{3}{2}$$

Example 9 *(**Calculation of square roots**) Let $a > 0$. We will construct a sequence $(a_n)_{n\in\mathbb{N}}$ such that*

$$\lim_{n\to\infty} a_n = \sqrt{a}$$

In other words, we would like to approximate $\sqrt{a}$ by a_n, provided that n is sufficiently large. To this end, we define recursively,

$$a_1 = 1 \text{ and } a_{n+1} = \frac{1}{2}\left(a_n + \frac{a}{a_n}\right)$$

Next, since

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{a}{a_n}\right) = \frac{1}{2}\left(\frac{a_n^2 + a}{a_n}\right)$$

we have

$$2a_{n+1}a_n = a_n^2 + a \Leftrightarrow a_n^2 + a - 2a_{n+1}a_n = 0$$

Treating $a_n^2 - 2a_{n+1}a_n + a$ as a quadratic in terms of a_n, we obtain the discriminant

$$\Delta = (-2a_{n+1})^2 - 4a = 4a_{n+1}^2 - 4a = 4\left(a_{n+1}^2 - a\right)$$

Since a_n is a solution for the quadratic equation

$$a_n^2 - 2a_{n+1}a_n + a = 0$$

then

$$4\left(a_{n+1}^2 - a\right) \geq 0 \Rightarrow a_{n+1}^2 \geq a.$$

Now, for any natural number n

$$\begin{aligned} a_n - a_{n+1} &= a_n - \frac{1}{2}\left(a_n + \frac{a}{a_n}\right) \\ &= a_n - \frac{1}{2}\left(\frac{a_n^2 + a}{a_n}\right) \\ &= \frac{2a_n^2 - a_n^2 - a}{2a_n} \\ &= \frac{a_n^2 - a}{2a_n} \geq 0 \end{aligned}$$

This shows that the given sequence is decreasing. As a result, by the Monotone Convergence theorem, the limit of our sequence exists. Let

$$s = \lim_{n\to\infty} a_n$$

Then

$$\begin{aligned}\lim_{n\to\infty} a_{n+1} &= \lim_{n\to\infty} \frac{1}{2}\left(a_n + \frac{a}{a_n}\right) = s \\ &= \frac{1}{2}\left(s + \frac{a}{s}\right)\end{aligned}$$

Thus,

$$s = \frac{1}{2}\left(s + \frac{a}{s}\right) = \frac{s^2 + a}{2s}$$

and

$$s^2 + a = 2s^2 \Leftrightarrow s^2 = a$$

Thus,

$$s = \sqrt{a}.$$