Section 3.3 Solutions

#4 Let $x_1 = 1$, $x_{n+1} = \sqrt{2+x_n}$ for $n \in \mathbb{N}$.

Then, $x_1 = 1$, $x_2 \approx 1.73$

$x_3 \approx 1.93$

$x_4 \approx 1.98$

$x_5 \approx 1.99$

We want to show that $(x_n)_{n\in\mathbb{N}}$ is an increasing sequence.

To this end, we show that for $x_{n+1} \geq x_n$ $\forall n \in \mathbb{N}$.

By induction, the base case holds for $n = 1$ since $x_2 = \sqrt{2+x_1} = \sqrt{3} > 1 = x_1$.

Now, assume that $x_{k+1} \geq x_k$ for some $k \in \mathbb{N}$, $k \geq 1$.

Then $x_{k+2} = \sqrt{2 + x_{k+1}} \geq \sqrt{2+x_k} = x_{k+1}$.

Consequently, $\forall n \in \mathbb{N}$, $x_{n+1} \geq x_n$.

Next, we shall show that $(x_n)_{n\in\mathbb{N}}$ is bounded. Indeed, we will prove that $\forall n \in \mathbb{N}$, $x_n \leq 2$. Clearly, the base case holds by definition. Next, suppose that $x_k \leq 2$ for some $k \in \mathbb{N}$. Then, $x_{k+1} = \sqrt{2+x_k} \leq \sqrt{2+2} = \sqrt{4} = 2$.

Therefore, $\forall n \in \mathbb{N}$, $x_n \leq 2$. Since $(x_n)_{n\in\mathbb{N}}$ is monotonic and bounded then it must be convergent by the monotone convergence theorem.

Next, let $x \in \mathbb{R}$ such that $\lim_{n\to\infty} x_n = x$. Then

$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \sqrt{2+x_n} = x = \sqrt{2+x}$. Next $x^2 = 2+x$ and

$x^2 - x - 2 = 0$. Using the quadratic formula $x = \frac{1+\sqrt{1+8}}{2} = \frac{1+\sqrt{9}}{2} = \frac{1+3}{2} = \frac{4}{2} = 2$.

Thus the recursively defined sequence $x_1 = 1$, $x_{n+1} = \sqrt{2+x_n}$ is convergent and $\lim_{n\to\infty} x_n = 2$.

#7 We want to prove by contradiction that this sequence is divergent.

By contradiction, suppose that $\lim\limits_{n\to\infty} x_n = L$ for some $L \in \mathbb{R}$.

Then $\lim\limits_{n\to\infty} x_{n+1} = \lim\limits_{n\to\infty} x_n + \frac{1}{x_n} = \lim\limits_{n\to\infty} x_n + \lim\limits_{n\to\infty} \frac{1}{x_n}$.

Thus $L = L + \frac{1}{L} \Leftrightarrow 0 = \frac{1}{L}$ and this is clearly absurd.

#11 First, $\forall n \in \mathbb{N}$,

$$x_{n+1} - x_n = \left(\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2} + \frac{1}{(n+1)^2}\right) - \left(\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2}\right)$$
$$= \frac{1}{(n+1)^2}.$$

Since the quantity $\left(\frac{1}{n+1}\right)^2 > 0$ for every $n \in \mathbb{N}$, it follows that $x_{n+1} - x_n > 0$. Thus $(x_n)_{n\in\mathbb{N}}$ is increasing.

Secondly, for any $n \in \mathbb{N}$,

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \leq 1 + \left(\frac{1}{1} - \frac{1}{2}\right) + \cdots + \left(\frac{1}{(n-1)^2} + \frac{1}{n^2}\right)$$
$$\leq 2 + \frac{1}{n^2}$$
$$\leq 2 + 1 = 3.$$

Thus the given sequence is monotone and bounded. As such, it is a convergent sequence.

(12a) First, note that $\left(1+\frac{1}{n}\right)^{n+1} = e^{(n+1)\ln\left(1+\frac{1}{n}\right)}$

Next, $\lim\limits_{n\to\infty} (n+1)\ln\left(1+\frac{1}{n}\right) = \lim\limits_{n\to\infty} \dfrac{\ln\left(1+\frac{1}{n}\right)}{\frac{1}{n+1}}$

(By L'Hopital's Rule, we have) $= \lim\limits_{n\to\infty} \dfrac{\frac{-1/n^2}{1+1/n}}{-\frac{1}{(n+1)^2}}$

$$= \lim_{n\to\infty} \frac{\frac{-1/n^2}{\frac{n+1}{n}}}{-\frac{1}{(n+1)^2}} = \lim_{n\to\infty} \frac{1}{n^2} \cdot \frac{n}{n+1} \cdot (n+1)^2$$

$$= \lim_{n\to\infty} \frac{n(n+1)^2}{n^2(n+1)} = \lim_{n\to\infty} \frac{n+1}{n} = 1.$$ Thus $\boxed{\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+1} = e^1 = e}$

(12b) $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{2n} = ?$ First, $\left(1+\frac{1}{n}\right)^{2n} = e^{2n\ln\left(1+\frac{1}{n}\right)}$.

Next, $\lim_{n\to\infty} 2n\ln\left(1+\frac{1}{n}\right) = \lim_{n\to\infty}\frac{\ln\left(1+\frac{1}{n}\right)}{(2n)^{-1}}$.

$$= \lim_{n\to\infty}\frac{\left(-1/n^2\right)/1+1/n}{-1\cdot(2n)^{-2}\,2} = \lim_{n\to\infty}\frac{\frac{\frac{1}{n^2}}{1+\frac{1}{n}}}{\frac{2}{(2n)^2}}$$

$$= \lim_{n\to\infty}\frac{1}{\cancel{n^2}}\cdot\frac{n}{n+1}\cdot\frac{^{2}\cancel{4}\cancel{n^2}}{\cancel{2}}$$

$$= \lim_{n\to\infty}\frac{2\cdot n}{n+1} = 2.\quad \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{2n} = e^2$$

(12c) $\lim_{n\to\infty}\left(1+\frac{1}{n+1}\right)^n = \lim_{n\to\infty} e^{n\ln\left(1+\frac{1}{n+1}\right)}$

Now $n\ln\left(1+\frac{1}{n+1}\right) = \frac{\ln\left(1+1/n+1\right)}{1/n} = \frac{\ln\left(\frac{n+2}{n+1}\right)}{1/n}$

$$= \frac{\ln(n+2)-\ln(n+1)}{n^{-1}}$$

$$\lim_{n\to\infty} n\cdot\ln\left(1+\frac{1}{n+1}\right) = \lim_{n\to\infty}\frac{1/n+2 - 1/n+1}{-1/n^2} = \lim_{n\to\infty}\frac{\frac{(n+1)-(n+2)}{(n+2)(n+1)}}{-1/n^2}$$

$$= \lim_{n\to\infty}\frac{\frac{-1}{(n+2)(n+1)}}{-1/n^2} = \lim_{n\to\infty}\frac{1}{(n+2)(n+1)}\cdot n^2 = \lim_{n\to\infty}\frac{n^2}{n^2+3n+2} = 1$$

$$\lim_{n\to\infty}\left(1+\frac{1}{n+1}\right)^n = e^{\lim_{n\to\infty} n\ln\left(1+\frac{1}{n+1}\right)}$$

$$= e^1 = e$$

(12d) $\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n = \lim_{n\to\infty} e^{n\ln(1-\frac{1}{n})}$.

Note that $n\cdot\ln(1-\frac{1}{n}) = n\ln\left(\frac{n-1}{n}\right)$

Next, $\lim_{n\to\infty} n\ln\left(\frac{n-1}{n}\right) = \lim_{n\to\infty}\dfrac{\ln(n-1)-\ln(n)}{n^{-1}} = \lim_{n\to\infty}\dfrac{\frac{1}{n-1}-\frac{1}{n}}{-\frac{1}{n^2}}$

$$= \lim_{n\to\infty}\dfrac{\frac{n-(n-1)}{(n-1)\,n}}{-1/n^2} = \lim_{n\to\infty} -\frac{1}{n\cdot(n-1)}\cdot\frac{n^2}{1} = -1$$

So $\boxed{\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n = e^{-1} = \frac{1}{e}}$

#14

Let $(s_n)_{n\in\mathbb{N}}$ be a sequence given by $s_1 = 1$, $s_{n+1} = \frac{1}{2}\left(s_n + \frac{5}{s_n}\right)$.
From Example 3.3.5, we know that $\lim_{n\to\infty} s_n = \sqrt{5}$. Next, we consider the following table.

n	s_n	$\lvert s_n-\sqrt{5}\rvert$
1	5	2.76393
2	3	0.763932
3	2.3333	0.0972654
4	2.2381	0.002027
5	2.23607	9.1814×10^{-7}

Thus $s_5 = \frac{2207}{987}$ is an approximation of $\sqrt{5}$, and the error made in this approximation is $\lvert s_5-\sqrt{5}\rvert < 10^{-5}$.