Section 3.2 (solutions)

#6 (a) $\lim\limits_{n\to\infty}\left(2+\frac{1}{n^2}\right) = \lim\limits_{n\to\infty}(2) + \lim\limits_{n\to\infty}\left(\frac{1}{n^2}\right)$ since the Limit of a sum, is a sum of limit.

$= 2+0$

$= \boxed{2}$

(b) $\lim\limits_{n\to\infty}\left|\frac{(-1)^n}{n+2}\right| = \lim\limits_{n\to\infty}\frac{1}{n+2} = \boxed{0}$. Since $\lim\limits_{n\to\infty} x_n = 0 \Leftrightarrow \lim\limits_{n\to\infty}|x_n| = 0$

then $\lim\limits_{n\to\infty}\frac{(-1)^n}{n+2} = 0$.

(c) $\lim\limits_{n\to\infty}\frac{\sqrt{n}-1}{\sqrt{n}+1} = ?$ $\quad \frac{\sqrt{n}-1}{\sqrt{n}+1} = \frac{(\sqrt{n}-1)(\sqrt{n}+1)}{(\sqrt{n}+1)(\sqrt{n}+1)} = \frac{n-1}{(\sqrt{n}+1)^2} = \frac{\cancel{n}\left(1-\frac{1}{n}\right)}{\cancel{n}\left(1+\frac{1}{\sqrt{n}}\right)^2}$

$= \frac{1-1/n}{(1+1/\sqrt{n})^2}$.

Then $\lim\limits_{n\to\infty}\frac{\sqrt{n}-1}{\sqrt{n}+1} = \lim\limits_{n\to\infty}\frac{1-1/n}{1+(1/\sqrt{n})^2} = \boxed{1}$

6d $\lim\limits_{n\to\infty}\frac{n+1}{n\sqrt{n}} = ?$

Given any $n\in\mathbb{N}$, $\frac{n+1}{n\sqrt{n}} \leqslant \frac{2n}{n\sqrt{n}} = \frac{2}{\sqrt{n}}$. Thus $\forall n\in\mathbb{N}$,

$0 \leq \frac{n+1}{n\sqrt{n}} \leq \frac{2}{\sqrt{n}}$. By the Squeeze theorem,

It follows that $\lim\limits_{n\to\infty}\frac{n+1}{n\sqrt{n}} = 0$ since $\lim\limits_{n\to\infty}\frac{2}{\sqrt{n}} = 0$ and $\lim\limits_{n\to\infty} 0 = 0$.

#9. Let $y_n = \sqrt{n+1} - \sqrt{n}$ for $n \in \mathbb{N}$. We want to show that $\sqrt{n}\, y_n$ is convergent.

Note that $\sqrt{n}\, y_n = \sqrt{n}\left(\sqrt{n+1} - \sqrt{n}\right) = \sqrt{n}\sqrt{n+1} - \sqrt{n}\sqrt{n} = \sqrt{n(n+1)} - n$

$$= \frac{\left(\sqrt{n(n+1)} - n\right)\left(\sqrt{n(n+1)} + n\right)}{\sqrt{n(n+1)} + n} = \frac{n(n+1) - n^2}{\sqrt{n(n+1)} + n}$$

$$= \frac{n^2 + n - n^2}{\sqrt{n(n+1)} + n} = \frac{n}{\sqrt{n(n+1)} + n} = \frac{n}{\sqrt{n^2+n} + n}$$

$$= \frac{\cancel{n} \cdot 1}{\cancel{n}\left(\sqrt{1 + \frac{1}{n}} + 1\right)} = \frac{1}{\sqrt{1 + 1/n} + 1}$$

Now since $\lim_{n\to\infty} \sqrt{1 + 1/n} + 1 = 1 + 1 = 2$. Thus $\lim_{n\to\infty} \frac{1}{\sqrt{1+1/n} + 1} = \frac{1}{2}$.

#13

Let $a>0, b>0$. We want to show that $\lim_{n\to\infty} \sqrt{(n+a)(n+b)} - n = \frac{a+b}{2}$.

__Proof__ Given $n \in \mathbb{N}$, $\sqrt{(n+a)(n+b)} - n = \frac{\left(\sqrt{(n+a)(n+b)} - n\right)\left(\sqrt{(n+a)(n+b)} + n\right)}{\sqrt{(n+a)(n+b)} + n}$

$$= \frac{(n+a)(n+b) - n^2}{\sqrt{(n+a)(n+b)} + n} = \frac{n^2 + bn + an + ab - n^2}{\sqrt{(n+a)(n+b)} + n} = \frac{bn + an + ab}{\sqrt{(n+a)(n+b)} + n}$$

$$= \frac{(a+b)n + ab}{\sqrt{n^2 + an + bn + ab} + n} = \frac{n\left((a+b) + \frac{ab}{n}\right)}{n\sqrt{1 + \frac{(a+b)}{n} + \frac{ab}{n^2}} + n} = \frac{\cancel{n}\left((a+b) + \frac{ab}{n}\right)}{\cancel{n}\left(\sqrt{1 + \frac{(a+b)}{n} + \frac{ab}{n^2}} + 1\right)}$$

Next, $\lim_{n\to n} \sqrt{(n+a)(n+b)} - n = \frac{\lim_{n\to\infty} (a+b) + \frac{ab}{n}}{\lim_{n\to\infty} \sqrt{1 + \frac{(a+b)}{n} + \frac{ab}{n^2}} + 1} = \frac{a+b}{1+1} = \frac{a+b}{2}$.

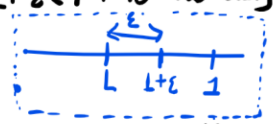
∎

#20

By assumption, $\forall \varepsilon>0, \exists N\in\mathbb{N}$ such that if $n>N$ then $|x_n^{1/n}-L|<\varepsilon$.

In other words given $\varepsilon>0$, $\exists N\in\mathbb{N}$ depending on ε such that if $n>N$ then $-\varepsilon < x_n^{1/n}-L<\varepsilon$. Now, note that $-\varepsilon < x_n^{1/n}-L<\varepsilon \iff L-\varepsilon < x_n^{1/n} < L+\varepsilon$ (*)

Next, since $L<1$, there exist $\varepsilon>0$ such that $L+\varepsilon<1$. To this end, It suffices to pick $\varepsilon>0$ such that $\varepsilon<1-L$



For such an $\varepsilon>0$, there exists $N\in\mathbb{N}$ such that as long as $n>N$ then $x_n^{1/n}<L+\varepsilon<1$. Next, Applying the nth Power to the inequality above, we obtain

$$(x_n^{1/n})^n < (L+\varepsilon)^n < 1^n$$. Next, letting $r=L+\varepsilon$. Then,

$0<x_n<r^n<1$.

Next, since $0<x_n<r^n$ for every $n\in\mathbb{N}$, it follows that

$$\lim_{n\to\infty} 0 < \lim_{n\to\infty} x_n < \lim_{n\to\infty} r^n.$$

Now since $0<r<1$, it follows that $\lim_{n\to\infty} r^n=0$ (See Ex 3.1.11 (b) on Page 60)

Finally $\lim_{n\to\infty} 0=0$. By the Squeeze theorem it follows that $\lim_{n\to\infty} x_n=0$ whenever $\lim_{n\to\infty} x_n^{1/n}=L<1$.