Section 2.3 (solutions)

(1) Let $A = \{x \in \mathbb{R} : x > 0\}$

(a) Yes, for example 0 is a lower bound for A since $\forall a \in A$, $a > 0$.

(b) No, suppose that A has an upperbound s. Note that $s > 1 > 0$ thus s is positive. Next $s+1$ is positive as well thus $s+1 \in A$. Since s is an upperbound of A, $s+1 \leq s \Rightarrow 1 < 0$ which is clearly absurd.

(c) $\inf A = 0$. To prove this we need to establish the following facts.

(c1) $\forall a \in A$, $a \geq 0$

(c2) let u be a lower bound of A. then $u \leq 0$

(c1) clearly holds by definition.

(c2) Suppose u is a lower bound of A and $u > 0$. Then $\frac{u}{2} > 0$ and as such, $\frac{u}{2} \in A$. But this implies that $u \leq \frac{u}{2}$ since u is a lowerbound for A. Consequently

$1 \leq \frac{1}{2} \Leftrightarrow 2 \leq 1 \Leftrightarrow 1 \leq 0$ which is absurd.

(d) Finally, since A does not have an upperbound it does not have a supremum.

(2) Let $B = \left\{ 1 - \frac{(-1)^n}{n} : n \in \mathbb{N} \right\}$
$= \left\{ 2, \frac{1}{2}, \frac{4}{3}, \frac{3}{4}, \frac{6}{5}, \cdots \right\}$

Claim $\sup B = 2$.

Proof Let $n \in \mathbb{N}$. First, note that $1 - \frac{(-1)^n}{n} \leq 2 \Leftrightarrow -\frac{(-1)^n}{n} \leq 1 \Leftrightarrow \frac{(-1)^{n+1}}{n} \leq 1$
$\Leftrightarrow (-1)^{n+1} \leq n$.

Now, since $(-1)^{n+1} \in \{1, -1\}$, clearly $(-1)^{n+1} \leq n \ \forall n \in \mathbb{N}$ and as a result, $1 - \frac{(-1)^n}{n} \leq 2$ for all $n \in \mathbb{N}$.

Thus 2 is an upperbound for B.

Next, we need to prove that 2 is the least upperbound of B.

Indeed, let $u \in \mathbb{R}$ such that u is an upperbound of B.

Then $\forall n \in \mathbb{N}$, $1 - \frac{(-1)^n}{n} \leq u$. Now for $n = 1$,
$1 - \frac{(-1)^n}{n} = 1 - (-1) = 2 \leq u$. Thus, we conclude that $2 = \sup B$

Claim $\inf B = \frac{1}{2}$.

Proof Let $n \in \mathbb{N}$. Note that $1 - \frac{(-1)^n}{n} \geq \frac{1}{2} \Leftrightarrow -\frac{(-1)^n}{n} \geq -1 + \frac{1}{2}$
$\Leftrightarrow -\frac{(-1)^n}{n} \geq -\frac{1}{2} \Leftrightarrow \frac{(-1)^n}{n} \leq \frac{1}{2} \Leftrightarrow (-1)^n \leq \frac{n}{2}$ (*). Next if

$n = 2K$ then (*) becomes $1 \leq \frac{2K}{2} = K$

If $n = 2K+1$ then (*) becomes $(-1) \leq \frac{2K+1}{2}$.

In light of the observations above, it is the case that $\forall n \in \mathbb{N}$, $1 - \frac{(-1)^n}{n} \geq \frac{1}{2}$.

Thus $\frac{1}{2}$ is a lowerbound for B. Next, if u is another lower bound for B, then $u \leq 1 - \frac{(-1)^n}{n} \ \forall n \in \mathbb{N}$. However if $n = 2$, $1 - \frac{(-1)^n}{n} = 1 - \frac{1}{2} = \frac{1}{2}$

Consequently $u \leq \frac{1}{2}$. Thus $\frac{1}{2}$ is the greatest lower bound for B and we conclude that $\inf(B) = \frac{1}{2}$.

(3) Let S be a nonempty subset of $\mathbb{R}$. Define $-S = \{-s : s \in S\}$.

We want to show that $\inf(S) = -\sup(-S)$. There 2 statements to prove

(a) $-\sup(-S)$ is a lowerbound of S

(b) $-\sup(-S)$ is the greatest lowerbound of S

To prove (a), let $s \in S$. Then $-s \in -S$ and $\sup(-S) \geq -s$. Thus $-\sup(-S) \leq s$.

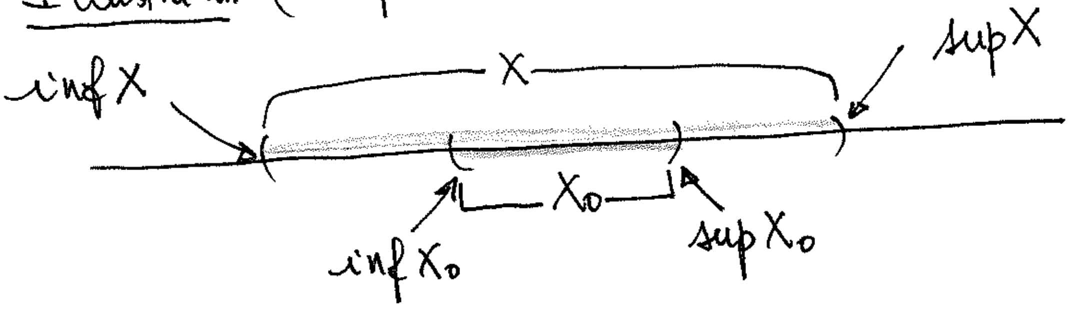
To prove (b), We need to establish the following: If y is a lowerbound for S then $y \leq -\sup(-S)$. Suppose otherwise. That is $y > -\sup(-S)$. Then $-y < \sup(-S)$. However $-y \in -S$ and it is not possible for $-y$ to be less than the supremum of a set it belongs to.

Since (a), (b) holds, it follows that $\inf(S) = -\sup(-S)$ $\square$

(4) Let $S \subseteq \mathbb{R}$ and $S \neq \phi$. Suppose that u is an upperbound for S. Next, let $t \in \mathbb{R}$ such that $t > u$. Suppose by contradiction that $t \in S$. Then by assumption (since u is an upperbound for S), $t \leq u < t$ which is clearly absurd.

Next, suppose that $t, u \in \mathbb{R}$ satisfying the following conditions: $t > u \Rightarrow t \notin S$ and $u \in \mathbb{R}$ is not an upperbound for S. Then by definition, there is $s \in S$ such that $s > u$. By assumption, it must be the case that $s \notin S$; contradicting the fact that s is an element of S. $\square$

(5) Illustration (scrap work: this is not a proof)



Proof of 5

For any $x_0 \in X_0$, $\inf X_0 \leq x_0 \leq \sup X_0$.

Thus, $\inf X_0 \leq \sup X_0$ (*)

Claim 1 $\inf X \leq \inf X_0$

Proof Let $x_0 \in X_0$. Since $X_0 \subseteq X$ then $x_0 \in X$. Consequently, $x_0 \geq \inf(X)$. Thus $\inf(X)$ is a lowerbound for X_0. However $\inf(X_0)$ is the greatest lowerbound of X_0. As a result, $\inf(X) \leq \inf(X_0)$. □

Claim 2 $\sup X_0 \leq \sup X$.

Proof Since $X_0 \leq X$, if $x_0 \in X_0$ then $x_0 \in X$ and $x_0 \leq \sup X$. Thus $\sup X$ is an upperbound for X_0. It follows that $\sup X_0 \leq \sup X$ since $\sup X_0$ is the least upperbound of X_0. □

Appealing to (*), Claim 1, Claim 2 it follows that

$$\inf X \leq \inf X_0 \leq \sup X_0 \leq \sup X.$$

(6) First, we need to prove that $\sup\{x, u\}$ is an upperbound for $X \cup \{u\}$.

Let $y \in X \cup \{u\}$. Then either $y \in S$ or $y = u$.

- If $y = u$ then $y \leq u$. Thus $y \leq \sup\{x, u\}$
- If $y \in S$ then $y \leq \sup(X) = x$. Thus $y \leq \sup\{x, u\}$

Secondly, we need to prove that $\sup\{x, u\}$ is the least upperbound for $X \cup \{u\}$. Let t be an upperbound for X. Then $u \leq t$. Moreover for any $y \in X$, $y \leq t$. Since X is the least upper bound for X, t has to be bigger than x. Thus $u \leq t$ and $x \leq t$ implies that $\sup\{x, u\} \leq t$.