Subsequences and the Bolzano-Weierstrass Theorem

- A **subsequence** of a sequence $(x_n)_{n\in\mathbb{N}}$ is a particular sequence whose terms are selected among those of the mother sequence $(x_n)_{n\in\mathbb{N}}$
- The study of subsequences is important because it provides vital information about the convergence of the parent sequence
- For example, we will see that a sequence which contains two subsequences converging to different limits is divergent
- An interesting result in this section (**Bolzano-Weierstrass Theorem**) states that every bounded sequence has a convergent subsequence
- **Plan**
  - Subsquences and properties
  - The Bolzano-Weierstrass Theorem

Definition 1 *Given a sequence* $(x_n)_{n\in\mathbb{N}}$ *of real numbers, let*

$$n_1 < n_2 < \cdots < n_k < \cdots <$$

be a strictly increasing sequence of natural numbers. Then the sequence given by

$$\left(x_{n_1}, x_{n_2}, \cdots, x_{n_k}, \cdots\right) = \left(x_{n_k}\right)_{k\in\mathbb{N}}$$

is called a ***subsequence*** *of* $(x_n)_{n\in\mathbb{N}}$.

This is a good place for you to pause the video for a moment.
Before you proceed, read this definition as many times as possible and try your best to unpack every aspect of this definition

Example 2

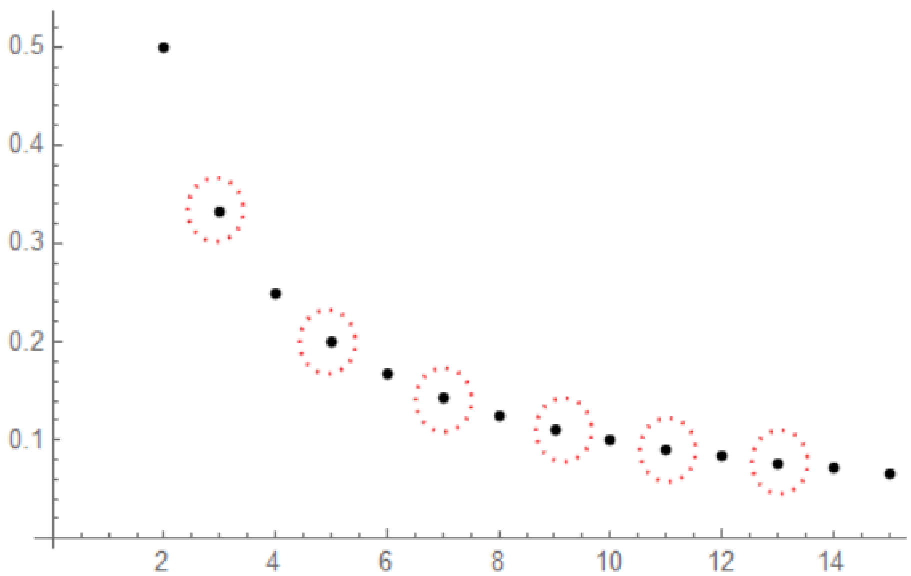
- *Let*

$$\left(\frac{1}{1}, \frac{1}{2}, \cdots, \frac{1}{k}, \cdots\right)$$

be a sequence. Then

$$\left(\frac{1}{2\cdot 1}, \frac{1}{2\cdot 2}, \cdots, \frac{1}{2\cdot k}, \cdots\right)$$

is a subsequence of the given sequence.



- *Let*

$$\left(2^1, 2^2, \cdots, 2^k, \cdots\right)$$

be a sequence. Then

$$\left(2^{2\cdot 1}, 2^{2\cdot 2}, \cdots, 2^{2\cdot k}, \cdots\right)$$

is a subsequence of the given sequence.

- *Let* $(x_n)_{n\in\mathbb{N}}$ *be a sequence such that*

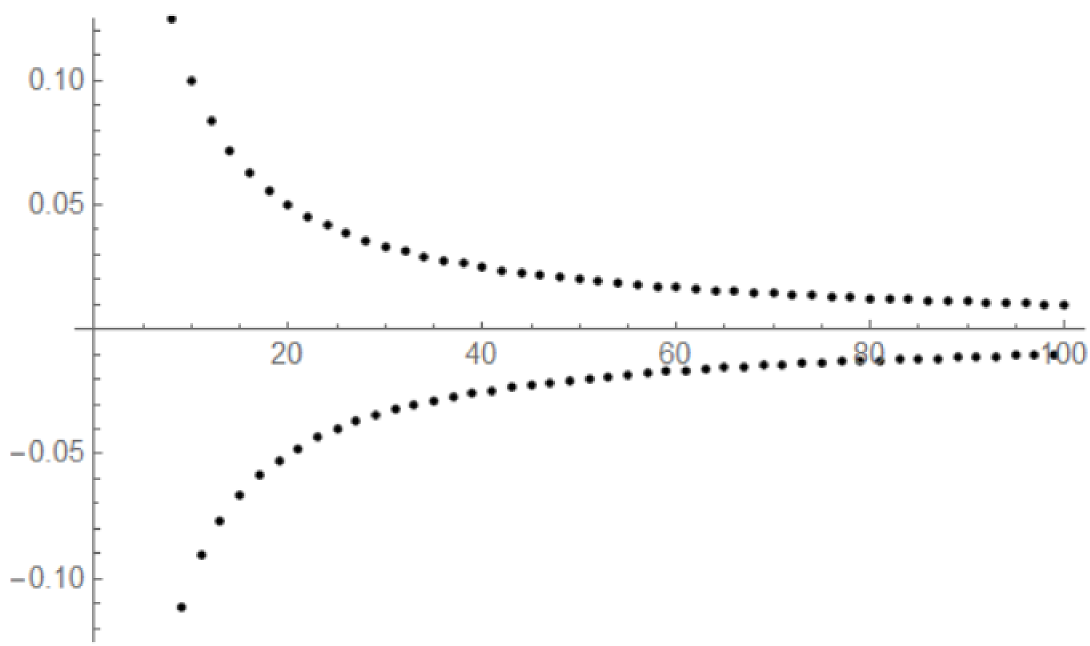
$$x_n = (-1)^n \frac{1}{n}.$$

Then $(x_{2k})_{k\in\mathbb{N}}$ *is a subsequence given by*

$$x_{2k} = (-1)^{2k} \frac{1}{2k} = \frac{1}{2k}.$$

Also, $(x_{2k+1})_{k\in\mathbb{N}}$ *is a subsequence given by*

$$x_{2k+1} = (-1)^{2k+1} \frac{1}{2k+1} = -\frac{1}{2k+1}.$$



- *Let*

$$\left(\frac{1}{1}, \frac{1}{2}, \cdots, \frac{1}{k}, \cdots\right)$$

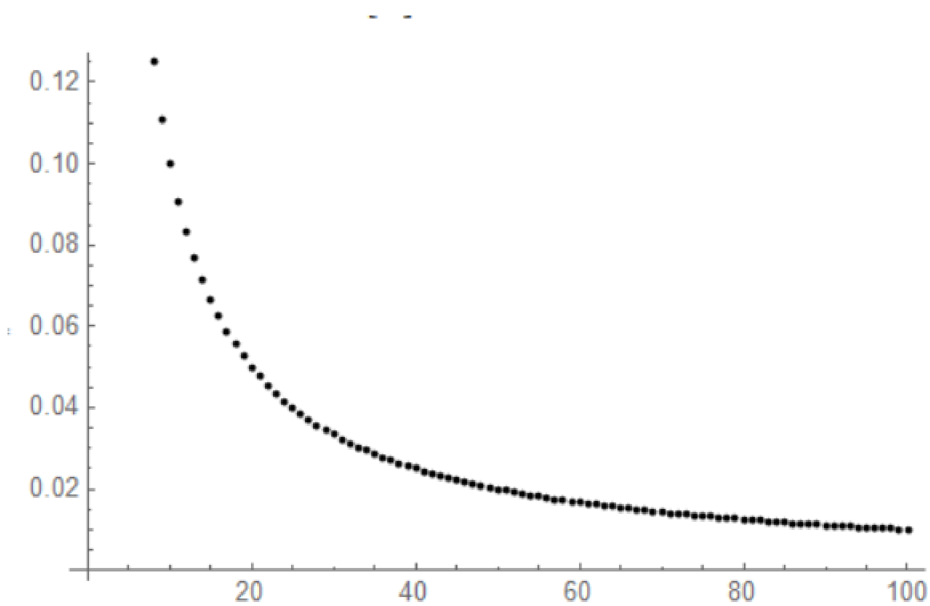
be a sequence. Then

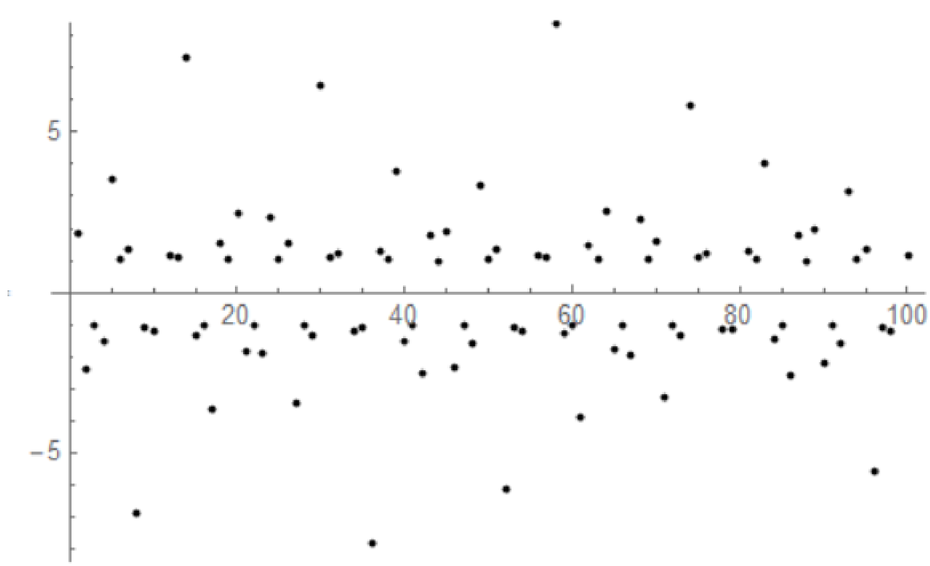
$$\left(\frac{1}{\cos(1)}, \frac{1}{\cos(2)}, \cdots, \frac{1}{\cos(k)}, \cdots\right)$$

is a not subsequence of the given sequence because

$$(\cos(k))_{k \in \mathbb{N}}$$

is not an increasing function.





Test your understanding: You should pause here, and (1) find an example of a sequence (2) write down an example of a subsequence of the sequence given in (1) and (2) find an example of a sequence which is not a subsequence of the sequence given in (1)

Theorem 3 *If* $(x_n)_{n\in\mathbb{N}}$ *is a sequence of real numbers which is convergent to* x *then any subsequence* $\left(x_{n_k}\right)_{k\in\mathbb{N}}$ *of* $(x_n)_{n\in\mathbb{N}}$ *is convergent to* x*.*

Proof. Let $\epsilon > 0$. By assumption, there exists a natural number N depending on ϵ such that if $n > N$ then

$$|x_n - x| < \epsilon.$$

I claim that since $(n_k)_{k\in\mathbb{N}}$ is an increasing sequence then it is true that

$$n_k \geq k \text{ for every natural number } k.$$

Indeed by induction we known that $n_1 \geq 1$ since n_1 is a natural number. Next, suppose that

$$n_\ell \geq \ell \text{ for some } \ell \in \mathbb{N}, \ell \geq 1.$$

Since $(n_k)_{k\in\mathbb{N}}$ is increasing then

$$n_{\ell+1} > n_\ell \text{ (strict inequality is important here !!!)}$$

Now by the inductive hypothesis,

$$n_\ell \geq \ell$$

and it follows that

$$n_{\ell+1} > \ell$$

Since $n_{\ell+1}, \ell$ are natural numbers

$$n_{\ell+1} \geq \ell + 1.$$

Moving forward, if $k > N$ then

$$n_k \geq k > N$$

and consequently,

$$\left|x_{n_k} - x\right| < \epsilon$$

Thus,

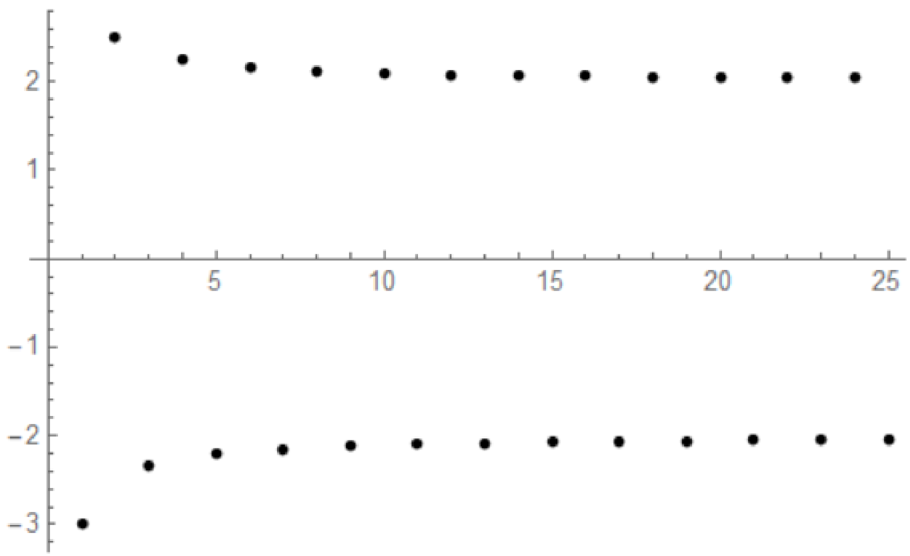
$$\lim_{k\to\infty} x_{n_k} = x$$

■

Corollary 4 *If* $(x_n)_{n\in\mathbb{N}}$ *is a sequence of real numbers admitting two subsequences which converge to different limits, then* $(x_n)_{n\in\mathbb{N}}$ *is divergent.*

Example 5 *Let us consider the following sequence*

$$a_n = (-1)^n \left(2 + \frac{1}{n}\right).$$



Then

$$a_{2k} = (-1)^{2k} \left(2 + \frac{1}{2k}\right) = 2 + \frac{1}{2k}$$

and

$$a_{2k+1} = (-1)^{2k+1} \left(2 + \frac{1}{2k+1}\right) = -\left(2 + \frac{1}{2k+1}\right)$$

Now note that

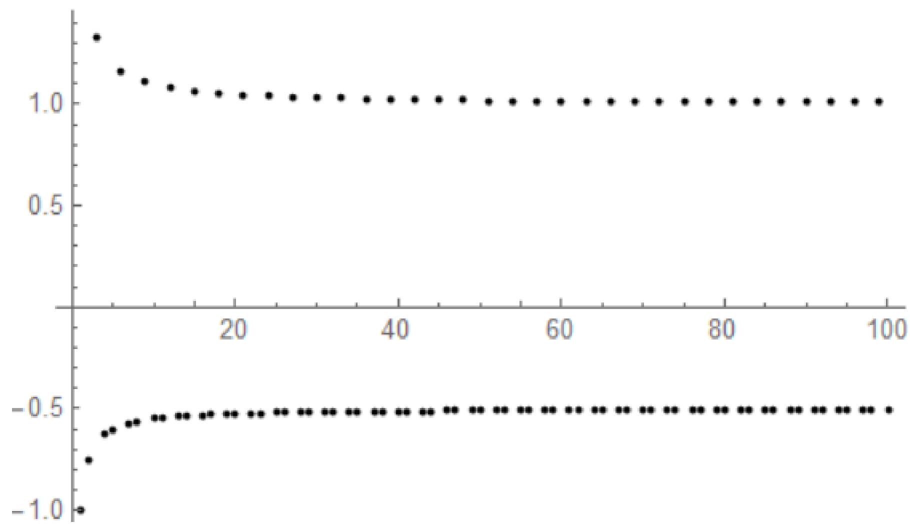
$$\lim_{k\to\infty} a_{2k} = \lim_{k\to\infty} 2 + \frac{1}{2k} = \lim_{k\to\infty} 2 + \lim_{k\to\infty} \frac{1}{2k} = 2$$

$$\lim_{k\to\infty} a_{2k+1} = \lim_{k\to\infty} -2 - \frac{1}{2k+1} = \lim_{k\to\infty} (-2) + \lim_{k\to\infty} \left(-\frac{1}{2k+1}\right) = -2$$

Thus, $(a_n)_{n\in\mathbb{N}}$ has two subsequences converging to two different limits. As such $(a_n)_{n\in\mathbb{N}}$ is a divergent sequence.

Example 6 *Let us consider the following sequence*

$$a_n = \cos\left(\frac{2\pi n}{3}\right)\left(1+\frac{1}{n}\right).$$



Then

$$\lim_{k\to\infty} a_{3k} = \lim_{k\to\infty} \cos\left(\frac{2\pi 3k}{3}\right)\left(1+\frac{1}{3k}\right)$$
$$= \lim_{k\to\infty}\left(1+\frac{1}{3k}\right) = 1$$

and

$$\lim_{k\to\infty} a_{3k+1} = \lim_{k\to\infty} \cos\left(\frac{2\pi\,(3k+1)}{3}\right)\left(1+\frac{1}{3k}\right)$$
$$= \lim_{k\to\infty} \cos\left(\frac{2\pi}{3}\right)\left(1+\frac{1}{3k}\right)$$
$$= -\frac{1}{2}\lim_{k\to\infty}\left(1+\frac{1}{3k}\right)$$
$$= -\frac{1}{2}$$

Therefore, the sequence $(a_n)_{n\in\mathbb{N}}$ *is a divergent sequence.*

Theorem 7 *Any sequence has a monotone subsequence*

Proof. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence. We will call a peak a term a_m of the sequence such that $a_m \geq a_n$ for all $n \geq m$. For the **first case**, let us suppose that $(a_n)_{n\in\mathbb{N}}$ has infinitely many peaks. We list those peaks by increasing order of their subscripts

$$a_{m_1}, a_{m_2}, \cdots, a_{m_k}, \cdots$$

and clearly by construction we have

$$a_{m_1} \geq a_{m_2} \geq \cdots \geq a_{m_k} \geq \cdots$$

Which is clearly a decreasing subsequence of $(a_n)_{n\in\mathbb{N}}$. For the **second case**, let us suppose that $(a_n)_{n\in\mathbb{N}}$ has a finite number of peaks (possibly none.) We list them in increasing order of their subscripts as follows

$$a_{m_1}, a_{m_2}, \cdots, a_{m_k}.$$

Since a_{m_k} is the last peak, then it is clear $a_{m_{k+1}}$ is not a peak. Thus, there is $n_1 > m_k + 1$ such that

$$a_{m_k+1} < a_{n_1}.$$

But a_{n_1} is not a peak. Thus, there is $n_2 > n_1$ such that $a_{n_1} < a_{n_2}$. We may then iterate this process indefinitely to construct an increasing sequence

$$a_{n_1} < a_{n_2} < a_{n_3} < \cdots < a_{n_k} < \cdots$$

■

Theorem 8 *(The Bolzano-Weierstrass Theorem) Any bounded sequence has a convergent subsequence.*

Proof. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence which is also bounded. Now from our previous result, we know that $(a_n)_{n\in\mathbb{N}}$ has a monotone subsequence say $\left(a_{n_k}\right)_{k\in\mathbb{N}}$. Since $\left(a_{n_k}\right)_{k\in\mathbb{N}}$ is a bounded sequence (as a subsequence of a bounded sequence) then $\left(a_{n_k}\right)_{k\in\mathbb{N}}$ must be a convergent subsequence of the parent sequence $(a_n)_{n\in\mathbb{N}}$. ■

Example 9 *Consider the sequence with terms* $\cos(n)$*. Since this sequence is bounded, then it has a convergent subsequence.*