Section 3.2 Limit Theorems

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Definition 1 *A sequence* $(x_n)_{n\in\mathbb{N}}$ *is said to be* ***bounded*** *if there exists a real number* $M > 0$ *such that*

$$|x_n| \leq M \text{ for every } n \in \mathbb{N}.$$

Example 2 *Let* $(x_n)_{n\in\mathbb{N}}$ *be a sequence with terms*

$$x_n = 1 + \frac{1}{n}.$$

For any given natural number n*, by the triangle inequality*

$$|x_n| = \left|1 + \frac{1}{n}\right| \leq |1| + \left|\frac{1}{n}\right| \leq 1 + 1 = 2.$$

Thus, the sequence $(x_n)_{n\in\mathbb{N}}$ *is* ***bounded****.*

Example 3 *Let* $(x_n)_{n\in\mathbb{N}}$ *be a sequence with terms*

$$x_n = n^2.$$

By the Archimedean Property, given a real number $M > 0$*, there exists a natural number* ℓ *such that*

$$\ell > \sqrt{M}.$$

Now squaring each side yields $\ell^2 > M$*. Consequently, given any positive real number* M*, there is a term* x_ℓ *of the sequence such that*

$$x_\ell > M.$$

Thus, the sequence $\left(n^2\right)_{n\in\mathbb{N}}$ *is* ***unbounded****.*

Theorem 4 *A convergent sequence of real numbers is bounded.*

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence and suppose that $(x_n)_{n\in\mathbb{N}}$ is convergent to some real number x. We want to show that $(x_n)_{n\in\mathbb{N}}$ must be a bounded sequence. Given $\epsilon = 1$, by assumption, there exists a natural number $N \in \mathbb{N}$ depending on 1 such that if $n > N$ then

$$|x_n - x| < 1.$$

To be pedantic, we may write N as $N(1)$, but we won't. Next,

$$|x_n| = |x_n + 0| = |x_n - x + x|$$

and by the Triangle Inequality, if $n > N$ we have

$$|x_n - x + x| \leq \underbrace{|x_n - x|}_{<1} + \ |x| < 1 + |x|.$$

Now, we write

$$\{|x_n| : n \in \mathbb{N}\} = \{|x_1|, \cdots, |x_N|\} \cup \{|x_n| : n > N\}$$

Let

$$M = \max\{|x_1|, \cdots, |x_N|, 1 + |x|\}.$$

For any $y \in \{|x_n| : n \in \mathbb{N}\}$ we have

$$y \leq M.$$

Thus, $(x_n)_{n\in\mathbb{N}}$ is a bounded sequence. ■

Remark 5 *The contrapositive of the statement if $(x_n)_{n\in\mathbb{N}}$ is a convergent sequence of real numbers then $(x_n)_{n\in\mathbb{N}}$ is bounded is if $(x_n)_{n\in\mathbb{N}}$ is unbounded then it is divergent is true.*

Example 6 *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence given by $x_n = \frac{1}{n^2}$. Since this sequence is convergent ($\lim \frac{1}{n^2} = 0$) then it is bounded.*

Example 7 *Becareful, there exist sequences which are bounded but NOT convergent. Here is an example, $((-1)^n)_{n\in\mathbb{N}}$ is a divergent sequence (Can you prove this, formally?) However,*

$$|(-1)^n| = 1 \leq 1$$

is a bounded sequence.

Problem 8 *If we know that $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are convergent, what can we say about the convergence of*

1. $(x_n + y_n)_{n\in\mathbb{N}}$
2. $(cx_n)_{n\in\mathbb{N}}$ *for some constant* c
3. $(x_n \cdot y_n)_{n\in\mathbb{N}}$
4. $\left(\dfrac{x_n}{y_n}\right)_{n\in\mathbb{N}}$ *if* $y_n \neq 0$ *for every natural number* n

Theorem 9 *If* $\lim_{n\to\infty} x_n = x$ *and* $\lim_{n\to\infty} y_n = y$ *then*

$$\lim_{n\to\infty} x_n + y_n = x + y.$$

Proof. We shall prove this result directly. To this end, suppose that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$. We want to prove that for any $\epsilon > 0$, there exists a natural number N such that if $n > N$ then

$$|x_n + y_n - (x + y)| < \epsilon.$$

Note that by the Triangle Inequality, we have

$$\begin{aligned} |x_n + y_n - (x + y)| &= |(x_n - x) + (y_n - y)| \\ &\leq |x_n - x| + |y_n - y| . \end{aligned}$$

Now to make $|x_n + y_n - (x - y)| < \epsilon$, it suffices to find n large enough so that

$$|x_n - x| < \frac{\epsilon}{2} \text{ and } |y_n - y| < \frac{\epsilon}{2}.$$

To this end, fix $\epsilon > 0$. Since $\lim_{n\to\infty} x_n = x$, there exists $N_1 \in \mathbb{N}$ such that if $n > N_1$ then

$$|x_n - x| < \frac{\epsilon}{2}.$$

Similarly, since $\lim_{n\to\infty} y_n = y$, there exists $N_2 \in \mathbb{N}$ such that if $n > N_2$ then

$$|y_n - y| < \frac{\epsilon}{2}.$$

To make sure that both conditions hold, put

$$N = \max\{N_1, N_2\}.$$

If $n > N$ then clearly

$$\begin{aligned} |x_n + y_n - (x + y)| &= |x_n - x + y_n - y| \\ &\leq |x_n - x| + |y_n - y| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

and this is the desired result. ■

Theorem 10 *If* $\lim_{n\to\infty} x_n = x$ *and* $\lim_{n\to\infty} y_n = y$ *then*

$$\lim_{n\to\infty} (x_n y_n) = xy.$$

Proof. We shall prove this result directly. To this end, suppose that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$. We want to prove that for any $\epsilon > 0$, there exists a natural number N such that if $n > N$ then

$$|x_n y_n - xy| < \epsilon.$$

We shall now introduce a nice little trick. We write

$$\begin{aligned} |x_n y_n - xy| &= \left| x_n y_n - xy - \overbrace{(x_n y - x_n y)}^{=0} \right| \\ &= |x_n y_n - x_n y + x_n y - xy| \\ &= |x_n (y_n - y) + (x_n - x) y| \end{aligned}$$

By the Triangle Inequality,

$$\begin{aligned} |x_n (y_n - y) + (x_n - x) y| &\leq |x_n (y_n - y)| + |(x_n - x) y| \\ &= |x_n| \cdot |y_n - y| + |x_n - x| \cdot |y| \,. \end{aligned}$$

Now to make

$$|x_n (y_n - y) + (x_n - x) y| < \epsilon$$

it suffices to make

$$|x_n| \cdot |y_n - y| < \frac{\epsilon}{2} \text{ and } |x_n - x| \cdot |y| < \frac{\epsilon}{2}$$

for sufficiently large natural numbers n. Let ϵ be a fixed positive real number. Since $(x_n)_n$ is convergent, then it is bounded. As such, there exists a real number $M > 0$ such that

$$|x_n| \leq M \text{ for every } n \in \mathbb{N}.$$

Thus,

$$\begin{aligned}\left|x_n\left(y_n-y\right)+\left(x_n-x\right)y\right| &\leq \left|x_n\left(y_n-y\right)\right|+\left|\left(x_n-x\right)y\right| \\ &= \left|x_n\right|\cdot\left|y_n-y\right|+\left|x_n-x\right|\cdot\left|y\right| \\ &\leq M\cdot\left|y_n-y\right|+\left|x_n-x\right|\cdot\left|y\right|.\end{aligned}$$

Setting

$$M_1=\max\left\{M,\left|y\right|\right\}$$

we have

$$\left|x_n\left(y_n-y\right)+\left(x_n-x\right)y\right|\leq M_1\cdot\left|y_n-y\right|+\left|x_n-x\right|\cdot M_1.$$

Now, since $\lim_{n\to\infty}y_n=y$ there exists $N_1\in\mathbb{N}$ such that if $n>N_1$ then

$$\left|y_n-y\right|<\frac{\epsilon}{2M_1}$$

Similarly, since $\lim_{n\to\infty}x_n=x$ there exists $N_2\in\mathbb{N}$ such that if $n>N_2$ then

$$\left|x_n-x\right|<\frac{\epsilon}{2M_1}.$$

Now, if $n>N=\max\left\{N_1,N_2\right\}$ we obtain

$$\begin{aligned}\left|x_n\left(y_n-y\right)+\left(x_n-x\right)y\right| &\leq \left(M_1\cdot\left|y_n-y\right|\right)+\left(\left|x_n-x\right|\cdot M_1\right) \\ &< M_1\left(\frac{\epsilon}{2M_1}\right)+\left(\frac{\epsilon}{2M_1}\right)M_1=\epsilon.\end{aligned}$$

Thus, in summary, given any $\epsilon>0$, there is $N\in\mathbb{N}$ such that if $n>N$ then

$$\left|x_ny_n-xy\right|=\left|x_n\left(y_n-y\right)+\left(x_n-x\right)y\right|<\epsilon$$

and this is the desired result. ■

Theorem 11 *If* $\lim_{n\to\infty} z_n = z$ *and* $z_n \neq 0$ *for all* $n \in \mathbb{N}$ *and* $z \neq 0$ *then*

$$\lim_{n\to\infty} \frac{1}{z_n} = \frac{1}{z}.$$

Proof. Suppose that $\lim_{n\to\infty} z_n = z$ and $z_n \neq 0$ for all $n \in \mathbb{N}$ and $z \neq 0$. We want to prove that for any $\epsilon > 0$, there exists a natural number N such that if $n > N$ then

$$\left|\frac{1}{z_n} - \frac{1}{z}\right| < \epsilon.$$

With straightforward calculations, we verify that

$$\left|\frac{1}{z_n} - \frac{1}{z}\right| = \left|\frac{z_n - z}{z_n z}\right| = \frac{1}{|z_n z|}\left|z_n - z\right| = \frac{1}{|z_n|}\frac{1}{|z|}\left|z_n - z\right|.$$

Since $\lim_{n\to\infty} z_n = z$, given $\xi > 0$ there exists a natural number N' depending on ξ such that $|z_n - z| < \xi$. Thus, for $n > N'$

$$||z_n| - |z|| < |z_n - z| < \xi.$$

Thus, $||z_n| - |z|| < \xi$ and

$$||z_n| - |z|| < \xi \Leftrightarrow -\xi < |z_n| - |z| < \xi \Leftrightarrow |z| - \xi < |z_n| < \xi + |z|$$

and as long as $n > N'$, we have

$$|z_n| > |z| - \xi.$$

Next, let $\xi = \frac{|z|}{2} > 0$. Then if $n > N'$, we have

$$|z_n| > |z| - \frac{|z|}{2} = \frac{|z|}{2}$$

and consequently,

$$\frac{1}{|z_n|} < \frac{2}{|z|}.$$

As such, for $n > N'$

$$\begin{aligned} \left|\frac{1}{z_n} - \frac{1}{z}\right| &= \left|\frac{z_n - z}{z_n z}\right| \\ &= \frac{1}{|z_n z|} \cdot |z_n - z| \\ &= \frac{1}{|z_n|}\frac{1}{|z|} \cdot |z_n - z| \\ &< \frac{2}{|z|}\frac{1}{|z|} \cdot |z_n - z| \\ &= \frac{2}{|z|^2} \cdot |z_n - z|. \end{aligned}$$

Next, since $\lim z_n = z$ there exists $N'' \in \mathbb{N}$ such that if $n > N''$ then

$$|z_n - z| < \epsilon \cdot \left(\frac{2}{|z|^2}\right)^{-1}.$$

Thus if $n > N = \max\{N', N''\}$ then

$$\left|\frac{1}{z_n} - \frac{1}{z}\right| < \frac{2}{|z|^2} \cdot |z_n - z| < \frac{2}{|z|^2}\left(\epsilon \cdot \left(\frac{2}{|z|^2}\right)^{-1}\right) = \frac{2}{|z|^2}\left(\frac{2}{|z|^2}\right)^{-1} \cdot \epsilon = \epsilon.$$

Therefore, $\lim_{n\to\infty}\left(\frac{1}{z_n}\right) = \frac{1}{z}$. ■

Corollary 12 *(The quotient rule for limit) If* $\lim_{n\to\infty} y_n = y$ *and* $y_n \neq 0, y \neq 0$ *for all* $n \in \mathbb{N}$ *and if* $\lim_{n\to\infty} x_n = x$ *then* $\lim_{n\to\infty} \frac{x_n}{y_n} = \frac{x}{y}$.

Proof. We have proved that

$$\lim_{n\to\infty} y_n = y \Rightarrow \lim_{n\to\infty} \frac{1}{y_n} = \frac{1}{y}.$$

Thus, by the product rule

$$\lim_{n\to\infty} x_n = x \text{ and } \lim_{n\to\infty} \left(\frac{1}{y_n}\right) = \frac{1}{y} \Rightarrow \lim_{n\to\infty} \left(\frac{x_n}{y_n}\right) = \frac{x}{y}.$$

■

Theorem 13 *If* $(x_n)_{n\in\mathbb{N}}$ *is convergent and if* $x_n \geq 0$ *for all* $n \in \mathbb{N}$ *then* $\lim_{n\to\infty} x_n = x \geq 0$.

Proof. Suppose by contradiction there exists a sequence $(x_n)_{n\in\mathbb{N}}$ which is convergent; $x_n \geq 0$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = x < 0$. Now, given $\epsilon = -x > 0$ there is a natural number N such that if $n > N$

$$|x_n - x| < \epsilon.$$

Now,

$$|x_n - x| < \epsilon \Leftrightarrow -\epsilon < x_n - x < \epsilon \Leftrightarrow x - \epsilon < x_n < x + \epsilon.$$

Thus, for $n > N$

$$x_n < x + \epsilon = x - x = 0$$

and this contradicts the assumption that $x_n \geq 0$ for all $n \in \mathbb{N}$. ■

Theorem 14 *If* $(x_n)_{n\in\mathbb{N}}$ *and* $(y_n)_{n\in\mathbb{N}}$ *are convergent sequences and if* $x_n \leq y_n$ *for all natural numbers* n *then* $\lim_{n\to\infty} x_n \leq \lim_{n\to\infty} y_n$

Proof. Let $(z_n)_{n\in\mathbb{N}}$ be a sequence given by $z_n = y_n - x_n$. By definition, $z_n \geq 0$ for all natural numbers. Next, by the previous results, $(z_n)_{n\in\mathbb{N}}$ is convergent and $\lim_{n\to\infty} z_n \geq 0$. Put

$$z = \overbrace{\lim_{n\to\infty} y_n}^{y} - \overbrace{\lim_{n\to\infty} x_n}^{x} = y - x \geq 0$$

By our previous result,

$$y \geq x.$$

■

Theorem 15 *Suppose that* $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$ *and* $(z_n)_{n\in\mathbb{N}}$ *are sequences of real numbers such that*

$$x_n \leq y_n \leq z_n \text{ for all } n \in \mathbb{N}$$

and suppose that $\lim x_n = \lim z_n$. *Then* $(y_n)_{n\in\mathbb{N}}$ *is convergent and*

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = \lim_{n\to\infty} z_n.$$

Proof. Let $w = \lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n$. For a fixed positive real number $\epsilon > 0$, there is a natural number N such that if $n > N$ then

$$|x_n - w| < \epsilon \text{ and } |z_n - w| < \epsilon$$

Now,

$$x_n \leq y_n \leq z_n \Rightarrow x_n - w \leq y_n - w \leq z_n - w$$

and under the assumption that $n > N$, we have

$$\begin{aligned} |x_n - w| < \epsilon &\Leftrightarrow -\epsilon < x_n - w < \epsilon \\ |z_n - w| < \epsilon &\Leftrightarrow -\epsilon < z_n - w < \epsilon. \end{aligned}$$

Thus,

$$-\epsilon < x_n - w \leq y_n - w \leq z_n - w < \epsilon$$

Consequently, $-\epsilon < y_n - w < \epsilon$ and it follows that $|y_n - w| < \epsilon$. Thus, $\lim_{n\to\infty} y_n = w$. ■