

The Cauchy Criterion

The Cauchy Criterion represents a way of identifying if a sequence is convergent without knowing the value of the limit in advance and without having information about its monotonicity.

Plan

- Definition of a Cauchy sequence, examples
- Cauchy criteria (a sequence is convergent if and only if it is Cauchy)

Definition 1 A sequence $(a_n)_{n \in \mathbb{N}}$ is said to be a Cauchy sequence if for any $\epsilon > 0$ there exists a natural number N such that if $m, n > N$ then

$$|a_n - a_m| < \epsilon$$

Claim 2 The sequence $(a_n)_{n \in \mathbb{N}}$ with

$$a_n = \frac{1}{n}$$

is a Cauchy sequence.

Discussion: We need to prove that for any $\epsilon > 0$ there exists a natural number N such that if $m, n > N$ then

$$\left| \frac{1}{n} - \frac{1}{m} \right| < \epsilon.$$

Note that

$$\left| \frac{1}{n} - \frac{1}{m} \right| \leq \frac{1}{n} + \frac{1}{m}$$

So to make

$$\frac{1}{n} + \frac{1}{m} < \epsilon$$

it is enough to find n, m sufficiently large so that

$$\frac{1}{n} < \frac{\epsilon}{2} \text{ and } \frac{1}{m} < \frac{\epsilon}{2}$$

To this end, it is enough to allow

$$n, m > \left(\frac{\epsilon}{2}\right)^{-1} = \frac{2}{\epsilon}.$$

Proof. Let $\epsilon > 0$. By the Archimedean Property, there exists a natural number N depending on ϵ such that

$$N > \frac{2}{\epsilon}$$

Next, suppose that $n > N$. Then

$$\frac{1}{n} < \frac{\epsilon}{2} \text{ and } \frac{1}{m} < \frac{\epsilon}{2}.$$

Consequently,

$$\left| \frac{1}{n} - \frac{1}{m} \right| \leq \frac{1}{n} + \frac{1}{m} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

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Claim 3 The sequence $(a_n)_{n \in \mathbb{N}}$ given by

$$a_n = 1 + (-1)^n$$

is not a Cauchy sequence.

Discussion: We need to prove that there exists $\epsilon > 0$ such that for every natural number, there exist $m, n > N$ and

$$|a_n - a_m| \geq \epsilon.$$

Note that

$$(a_n)_{n \in \mathbb{N}} = (0, 2, 0, 2, 0, 2, \dots)$$

Next, given natural numbers n, m

$$|a_n - a_m| = |1 + (-1)^n - 1 - (-1)^m| = |(-1)^n - (-1)^m|.$$

Moreover,

$$|(-1)^n - (-1)^m| \in \{0, 2\}$$

In fact, if n is even and $m = n + 1$ then

$$|(-1)^n - (-1)^m| = |1 + 1| = 2$$

Otherwise if n is odd and $m = n + 1$ then

$$|(-1)^n - (-1)^m| = |-1 - 1| = 2$$

Thus by selecting $\epsilon = 1$, no matter how large N is, we can find $n, m > N$ such that

$$|(-1)^n - (-1)^m| > 1$$

In other words, we cannot make the distance between consecutive terms arbitrary small.

Proof. Let $\epsilon = 1$. Given any natural number N , we note that

$$\left| (-1)^{N+1} - (-1)^{N+2} \right| = 2 \geq 1.$$

As a result, the sequence

$$(1 + (-1)^n)_{n \in \mathbb{N}}$$

is not a Cauchy sequence. ■

Theorem 4 *Any Cauchy sequence is necessarily a bounded sequence.*

Proof. Let $(a_n)_{n \in \mathbb{N}}$ be a Cauchy sequence. Fix $\epsilon = 1$. There exists a natural number N depending on 1 such that for any $n > N$ we have

$$|a_n - a_N| < 1$$

Observe that

$$\begin{aligned} |a_n - a_N| < 1 &\Leftrightarrow -1 < a_n - a_N < 1 \\ &\Leftrightarrow -1 + a_N < a_n < 1 + a_N \\ &\Leftrightarrow a_n \in (-1 + a_N, 1 + a_N) \end{aligned}$$

This means that

$$\{a_n : n > N\}$$

is a bounded set. In fact $1 + a_N$ is an upperbound for such a set. But

$$\{a_n : 1 \leq n \leq N\}$$

is also a bounded set since it is a finite set. Thus,

$$\{a_n : n \in \mathbb{N}\}$$

is a bounded set since it is a union of two bounded sets (can you prove this fact?) ■

Theorem 5 (Cauchy-Criterion) *A sequence is convergent if and only if it is Cauchy*

Proof. Assume that $(a_n)_{n \in \mathbb{N}}$ is a convergent sequence. Moreover, suppose there exists some real number $a > 0$ such that

$$\lim_{n \rightarrow \infty} a_n = a.$$

Observe that for any given natural number n

$$\begin{aligned} |a_n - a_m| &= |a_n - a_m - a + a| \\ &\leq |a_n - a + a - a_m| \\ \text{(Triangle Inequality)} &\leq |a_n - a| + |a_m - a|. \end{aligned}$$

Now to make

$$|a_n - a_m| < \epsilon$$

it suffices to make

$$|a_n - a| < \frac{\epsilon}{2} \text{ and } |a_m - a| < \frac{\epsilon}{2}.$$

By assumption there exists a natural number N depending on ϵ such that if $n > N$ then $|a_n - a| < \frac{\epsilon}{2}$. Fixing such an N , as long as both n, m are larger than N we obtain the desired result:

$$|a_n - a_m| \leq |a_n - a| + |a_m - a| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus, the sequence $(a_n)_{n \in \mathbb{N}}$ is Cauchy. Regarding the converse, let us assume that $(a_n)_{n \in \mathbb{N}}$ is a Cauchy sequence. From our previous result, $(a_n)_{n \in \mathbb{N}}$ is a bounded sequence. Next, by the Bolzano-Weierstrass theorem, $(a_n)_{n \in \mathbb{N}}$ has a convergent subsequence $(a_{n_k})_{k \in \mathbb{N}}$. Let a be a real number such that

$$\lim_{k \rightarrow \infty} a_{n_k} = a.$$

We aim to show that $(a_n)_{n \in \mathbb{N}}$ converges to a as well. Let ϵ be a fixed positive real number. Then there exists a natural number N

depending on ϵ such that

$$|a_{n_k} - a| < \frac{\epsilon}{2} \quad (*)$$

provided that $k > N$. Now, since $(a_n)_{n \in \mathbb{N}}$ is Cauchy, there exists $M \in \mathbb{N}$ depending on ϵ such that if $n, m > M$ then

$$|a_n - a_m| < \frac{\epsilon}{2} \quad (**)$$

Next, for $k > \max \{N, M\}$, we remark that $n_k \geq k$ and

$$\begin{aligned} |a_k - a| &= |a_k - a - a_{n_k} + a_{n_k}| \\ \text{(Triangle Inequality)} &\leq |a_k - a_{n_k}| + |a_{n_k} - a| \\ \text{By } (*) \text{ and } (**) &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

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Example 6 *The harmonic series*

$$a_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} \text{ is divergent.}$$

Proof. Let

$$a_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$$

and

$$a_{2n} = 1 + \frac{1}{2} + \cdots + \frac{1}{2n}$$

Then

$$\begin{aligned} a_{2n} - a_n &= \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n}\right) - \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) \\ &= \underbrace{\frac{1}{n+1} + \cdots + \frac{1}{2n}}_{n \text{ times}} \\ &\geq \frac{1}{2n} + \cdots + \frac{1}{2n} \\ &= \frac{n}{2n} = \frac{1}{2} \end{aligned}$$

Thus, for $\epsilon = 1/2$, for any natural number $N - 1$

$$|a_{2N} - a_N| \geq \frac{1}{2}$$

We conclude that $(a_n)_{n \in \mathbb{N}}$ is not Cauchy and thus, it is not convergent. ■

Example 7 Let $(a_n)_{n \in \mathbb{N}}$ be a sequence of natural numbers given by

$$a_1 = 1, a_2 = 2 \text{ and } a_n = \frac{a_{n-1} + a_{n-2}}{2} \text{ for } n \geq 3$$

Prove that this sequence is convergent.

Proof. I would like to prove that

$$|a_{n+1} - a_n| = \frac{1}{2^{n-1}} \text{ for any natural number } n.$$

First, for $n = 1$, we verify that

$$|a_2 - a_1| = 1 = \frac{1}{2^0}.$$

Thus, the base case holds. Next, assume that

$$|a_{k+1} - a_k| = \frac{1}{2^{k-1}} \text{ for some natural number } k.$$

Then

$$\begin{aligned} |a_{k+2} - a_{k+1}| &= \left| \frac{a_{k+1} + a_k}{2} - a_{k+1} \right| \\ &= \left| \frac{a_k - a_{k+1}}{2} \right| \\ \text{(By the inductive hypothesis)} &= \frac{1}{2} \frac{1}{2^{k-1}} \\ &= \frac{1}{2^k} \end{aligned}$$

Next, for any natural number p we have

$$\begin{aligned} |a_{n+p} - a_n| &= |(a_{n+p} - a_{n+p-1}) + (a_{n+p-1} - a_{n+p-2}) + \cdots + (a_{n+1} - a_n)| \\ \text{(Triangle Inequality)} &\leq |a_{n+p} - a_{n+p-1}| + |a_{n+p-1} - a_{n+p-2}| + \cdots + |a_{n+1} - a_n| \\ &\leq \frac{1}{2^{n+p-2}} + \frac{1}{2^{n+p-3}} + \cdots + \frac{1}{2^{n-1}} \\ &= \frac{1}{2^{n-1}} \left(\frac{1}{2^{p-1}} + \cdots + \frac{1}{2} + 1 \right) \\ &< \frac{1}{2^{n-1}} \cdot \left(\sum_{k=0}^{\infty} \frac{1}{2^k} \right) \\ &= \frac{2}{2^{n-1}} \end{aligned}$$

Let $\epsilon > 0$. We can find n to be sufficiently large so that

$$\frac{2}{2^{n-1}} < \epsilon.$$

Indeed,

$$\frac{2}{2^{n-1}} = \frac{1}{2^{n-2}} \leq \frac{10}{n}$$

and for $n > \frac{10}{\epsilon}$, we have

$$|a_{n+p} - a_n| < \frac{1}{2^{n-2}} \leq \frac{10}{n} < \epsilon$$

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