

Lecture notes on the Plancherel formula for the Heisenberg group

H is the heisenberg group with lie algebra \mathfrak{h}

$$H = \left\{ \begin{bmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{bmatrix} : x, y, z \in \mathbb{R} \right\}$$

$$\mathfrak{h} = \left\{ \begin{bmatrix} 0 & b & c \\ 0 & 0 & a \\ 0 & 0 & 0 \end{bmatrix} : a, b, c \in \mathbb{R} \right\} = \mathbb{R}^3$$

It is well known that the set of all irreducible representations of H is parametrized by $\mathbb{R} \setminus \{0\}$ and that the dual group of H ,

$$\widehat{H} = \{ \pi_\lambda : \pi_\lambda = \text{ind}_D^H \chi_\lambda, \lambda \in \Omega \}, \text{ where}$$

$$D = \left\{ \exp \begin{bmatrix} 0 & b & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & b & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} : a, b \in \mathbb{R} \right\}$$

$$\Omega = \left\{ l \in \mathfrak{h}^* : l \left(\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) \neq 0 \right\}$$

It is not hard to see that $\Omega = \{(\lambda, \beta, \alpha) : \lambda \neq 0\}$

Proposition 1

$$\left(\text{ind}_D^H \chi_\lambda \left(\begin{bmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{bmatrix} F \right) \right) \left(\begin{bmatrix} 1 & y_1 & z_1 \\ 0 & 1 & x_1 \\ 0 & 0 & 1 \end{bmatrix} \right)$$

$$= e^{-i\lambda(z_1 - z - yx_1 + xy)} F \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & x_1 - x \\ 0 & 0 & 1 \end{bmatrix} \right)$$

Proof. The representation $\text{ind}_D^H \chi_\lambda = \pi_\lambda$ acts on the hilbert completion of

$$\mathbf{F}_\lambda = \left\{ F : H \rightarrow \mathbb{C} : F(x\xi) = e^{-i\lambda(\xi)} F(x), x \in N/D, \xi \in D \right\}$$

by left translations.
$$\begin{aligned} & \left(\pi_\lambda \left(\begin{bmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{bmatrix} \right) F \right) \left(\begin{bmatrix} 1 & y_1 & z_1 \\ 0 & 1 & x_1 \\ 0 & 0 & 1 \end{bmatrix} \right) \\ &= F \left(\begin{bmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & y_1 & z_1 \\ 0 & 1 & x_1 \\ 0 & 0 & 1 \end{bmatrix} \right) \\ &= F \left(\begin{bmatrix} 1 & y_1 - y & z_1 - z - yx_1 + xy \\ 0 & 1 & x_1 - x \\ 0 & 0 & 1 \end{bmatrix} \right) \\ &= F \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & x_1 - x \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & y_1 - y & z_1 - z - yx_1 + xy \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \\ &= \chi_\lambda \left(\begin{bmatrix} 1 & y_1 - y & z_1 - z - yx_1 + xy \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) F \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & x_1 - x \\ 0 & 0 & 1 \end{bmatrix} \right) \\ &= e^{-i\lambda(z_1 - z - yx_1 + xy)} F \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & x_1 - x \\ 0 & 0 & 1 \end{bmatrix} \right) \end{aligned}$$

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Proposition 2 *Parametrizing the dual group of the heisenberg group by $\mathbb{R} \setminus \{0\}$, the lebesgue measure on \widehat{H} is $d\lambda$, and the plancherel measure is $|\lambda| d\lambda$*

Proof.

$$\begin{aligned} (\pi_\lambda(F)\phi)(t) &= \iiint F \left(\begin{bmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{bmatrix} \right) \left(\pi_\lambda \left(\begin{bmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{bmatrix} \right) \phi \right) (t) d \left(\begin{bmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{bmatrix} \right) \\ &= \iiint F \left(\begin{bmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{bmatrix} \right) e^{-i\lambda(z_1 - z - yx_1 + xy)} \phi \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & t - x \\ 0 & 0 & 1 \end{bmatrix} \right) d \left(\begin{bmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{bmatrix} \right) \\ &= \int \int \int F(z, y, x) e^{-i\lambda(z_1 - z - yx_1 + xy)} \phi(t - x) dx dy dz \\ &= \int \left(\int \int F(z, y, t - x) e^{-i\lambda(z_1 - z - yx_1 + xy)} \phi(x) dy dz \right) dx \end{aligned}$$

The operator $\pi_\lambda(F)$ has for kernel $K(t, x) = \int \int F(z, y, t - x) e^{-i\lambda(z_1 - z - yx_1 + xy)} dy dz$.

Computing the hilbert schmidt norm of the operator $\pi_\lambda(F)$,

$$\|\pi_\lambda(F)\|_{HS}^2 = \int \int |K(t, x)|^2 dt dx$$

Denoting the partial fourier transforms \mathcal{F}_x , \mathcal{F}_y , and \mathcal{F}_z resp in x , y and z directions, we observe that

$$\begin{aligned}
& \int \int F(z, y, t-x) e^{-i\lambda(z_1 - z - yx_1 + xy)} dy dz \\
= & \int \int F(z, y, t-x) e^{-i\lambda z_1} e^{i\lambda z} e^{i\lambda y(x-x_1)} dy dz \\
= & e^{-i\lambda z_1} \int \int F(z, y, t-x) e^{i\lambda z} e^{i\lambda y x} e^{-i\lambda y x_1} dy dz \\
= & \mathcal{F}_x \circ \mathcal{F}_y \circ \mathcal{F}_z \circ F(\lambda, \lambda x, u)
\end{aligned}$$

Put $G(\lambda, x, u) = F(\lambda, \lambda x, u)$, and applying the parseval equality,

$$\begin{aligned}
\|\pi_\lambda(F)\|_{HS}^2 &= \|K\|_{L^2(\mathbb{R}^2)}^2 \\
&= \|\mathcal{F}_x \circ \mathcal{F}_y \circ \mathcal{F}_z \circ G\|_{L^2(\mathbb{R}^2)}^2 \\
&= \|\mathcal{F}_y \circ \mathcal{F}_z \circ G\|_{L^2(\mathbb{R}^2)}^2 \\
&= \|\mathcal{F}_z \circ G\|_{L^2(\mathbb{R}^2)}^2 \\
&= \|G\|_{L^2(\mathbb{R}^2)}^2 \\
&= |\lambda|^{-1} \|F\|_{L^2(\mathbb{R}^2)}^2 \cdot \\
|\lambda| \|\pi_\lambda(F)\|_{HS}^2 &= \|F\|_{L^2(\mathbb{R}^2)}^2
\end{aligned}$$

Next,

$$\begin{aligned}
\int \|\pi_\lambda(F)\|_{HS}^2 |\lambda| d\lambda &= \int \|F\|_{L^2(\mathbb{R}^2)}^2 d\lambda \\
&= \|F\|_H^2.
\end{aligned}$$

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