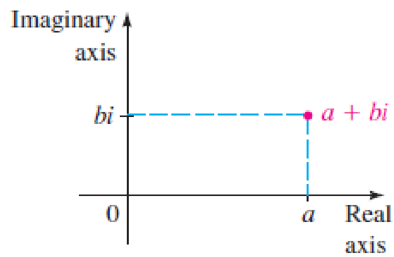
7-5 Complex Numbers and De Moivre's Theorem

Graphing Complex Numbers

To graph real numbers or sets of real numbers, we have been using the number line, which has just one dimension. Complex numbers, however, have two components: a real part and an imaginary part. This suggests that we need two axes to graph complex numbers: one for the real part and one for the imaginary part. We call these the **real axis** and the **imaginary axis**, respectively. The plane determined by these two axes is called the **complex plane**. To graph the complex number $a + bi$, we plot the ordered pair of numbers (a, b) in this plane, as indicated in Figure 1.



The **modulus** (or **absolute value**) of the complex number $z = a + bi$ is

$$|z| = \sqrt{a^2 + b^2}$$

Example 3 Calculating the Modulus

Find the moduli of the complex numbers $3 + 4i$ and $8 - 5i$.

Solution

$$|3 + 4i| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$$

$$|8 - 5i| = \sqrt{8^2 + (-5)^2} = \sqrt{89}$$

Plotting in the Complex Plane

Plot the following complex numbers in a complex plane:

$$A = 2 + 3i \qquad B = -3 + 5i \qquad C = -4 \qquad D = -3i$$

Polar Form of Complex Numbers

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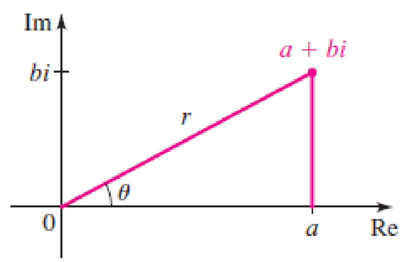
A complex number $z = a + bi$ has the **polar form** (or **trigonometric form**)

$$z = r(\cos\theta + i\sin\theta)$$

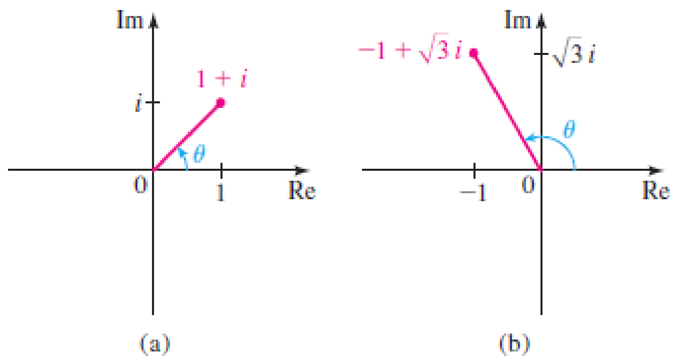
where $r = |z| = \sqrt{a^2 + b^2}$ and $\tan\theta = b/a$. The number r is the **modulus** of z, and θ is an **argument** of z.



Example 5 Writing Complex Numbers in Polar Form

Write each complex number in trigonometric form.

(a) $1 + i$ (b) $-1 + \sqrt{3}i$


Im
1 + i
i
θ
0
1
Re
(a)
Im
−1 + √3 i
√3 i
θ
−1
0
Re
(b)

$$\tan\theta = \tfrac{1}{1} = 1$$
$$\theta = \tfrac{\pi}{4}$$

(a) An argument is $\theta = \pi/4$ and $r = \sqrt{1 + 1} = \sqrt{2}$. Thus

$$1 + i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

$$\tan\theta = \frac{\sqrt{3}}{-1} = -\sqrt{3}$$

$$\theta = \tfrac{2\pi}{3}$$

(b) An argument is $\theta = 2\pi/3$ and $r = \sqrt{1 + 3} = 2$. Thus

$$-1 + \sqrt{3}i = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$$

Multiplication and Division of Complex Numbers

If the two complex numbers z_1 and z_2 have the polar forms

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \qquad \text{and} \qquad z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

then

$$z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)] \qquad \text{Multiplication}$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)] \qquad (z_2 \neq 0) \qquad \text{Division}$$

Euler form

$$
\begin{aligned}
e^{i\theta} &= \cos\theta + i\sin\theta \\
re^{i\theta} &= r(\cos\theta + i\sin\theta)
\end{aligned}
$$

If $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$, then

1. $z_1 z_2 = r_1 e^{i\theta_1} r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1 + \theta_2)}$

2. $\dfrac{z_1}{z_2} = \dfrac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \dfrac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}, \qquad r_2 \neq 0$

Products and Quotients

If $z_1 = 8e^{45°i}$ and $z_2 = 2e^{30°i}$, find

(A) $z_1 z_2$ (B) z_1/z_2

DeMoivre's Theorem

If $z = r(\cos\theta + i\sin\theta)$, then for any integer n

$$z^n = r^n(\cos n\theta + i\sin n\theta)$$

Find $\left(\frac{1}{2} + \frac{1}{2}i\right)^{10}$.

Solution Since $\frac{1}{2} + \frac{1}{2}i = \frac{1}{2}(1 + i)$, it follows from Example 5(a) that

$$\frac{1}{2} + \frac{1}{2}i = \frac{\sqrt{2}}{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

$$\left(\frac{1}{2} + \frac{1}{2}i\right)^{10} = \left(\frac{\sqrt{2}}{2}\right)^{10}\left(\cos\frac{10\pi}{4} + i\sin\frac{10\pi}{4}\right)$$

$$= \frac{2^5}{2^{10}}\left(\cos\frac{5\pi}{2} + i\sin\frac{5\pi}{2}\right) = \frac{1}{32}i$$