6-4 Product–Sum and Sum–Product Identities

1 Express Products as Sums

THEOREM

Product-to-Sum Formulas

$$\sin \alpha \sin \beta = \frac{1}{2}[\cos(\alpha - \beta) - \cos(\alpha + \beta)]$$

$$\cos \alpha \cos \beta = \frac{1}{2}[\cos(\alpha - \beta) + \cos(\alpha + \beta)]$$

$$\sin \alpha \cos \beta = \frac{1}{2}[\sin(\alpha + \beta) + \sin(\alpha - \beta)]$$

EXAMPLE

Expressing Products as Sums

Express each of the following products as a sum containing only sines or cosines.

(a) $\sin(3\theta)\sin(7\theta) = \frac{1}{2}\left[\cos(3\theta - 7\theta) - \cos(3\theta + 7\theta)\right]$

$$= \frac{1}{2}\left[\cos(-4\theta) - \cos(10\theta)\right] = \frac{1}{2}\left[\cos(4\theta) - \cos(10\theta)\right]$$

(b) $\cos\theta\cos(5\theta) = \frac{1}{2}\left[\cos(\theta - 5\theta) + \cos(\theta + 5\theta)\right]$

$$= \frac{1}{2}\left[\cos(-4\theta) + \cos(6\theta)\right] = \frac{1}{2}\left[\cos(4\theta) + \cos(6\theta)\right]$$

(c) $\sin(2\theta)\cos(7\theta) = \frac{1}{2}\left[\sin(2\theta + 7\theta) + \sin(2\theta - 7\theta)\right]$

$$= \frac{1}{2}\left[\sin(9\theta) + \sin(-5\theta)\right]$$

$$= \frac{1}{2}\left[\sin(9\theta) - \sin(5\theta)\right]$$

$$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha - \beta) - \cos(\alpha + \beta)]$$

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha - \beta) + \cos(\alpha + \beta)]$$

$$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha + \beta) + \sin(\alpha - \beta)]$$

2 Express Sums as Products

THEOREM

Sum-to-Product Formulas

$$\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

$$\sin\alpha - \sin\beta = 2\sin\frac{\alpha-\beta}{2}\cos\frac{\alpha+\beta}{2}$$

$$\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

$$\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$

EXAMPLE **Expressing Sums (or Differences) as a Product**

Express each sum or difference as a product of sines and/or cosines.

(a) $$\sin(4\theta) - \sin(6\theta) = 2\sin\frac{4\theta - 6\theta}{2}\cos\frac{4\theta + 6\theta}{2}$$

$$= 2\sin\frac{-2\theta}{2}\cos\frac{10\theta}{2} = -2\sin\theta\cos 5\theta$$

(b) $$\cos(2\theta) + \cos(8\theta) = 2\cos\frac{2\theta + 8\theta}{2}\cos\frac{2\theta - 8\theta}{2}$$

$$= 2\cos\frac{10\theta}{2}\cos\frac{-6\theta}{2}$$

$$= 2\cos 5\theta\cos 3\theta$$

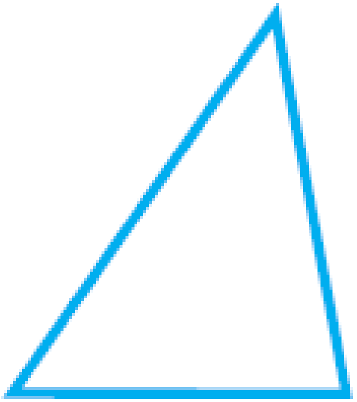
$$\sin\alpha + \sin\beta = 2\sin\frac{\alpha + \beta}{2}\cos\frac{\alpha - \beta}{2}$$

$$\sin\alpha - \sin\beta = 2\sin\frac{\alpha - \beta}{2}\cos\frac{\alpha + \beta}{2}$$

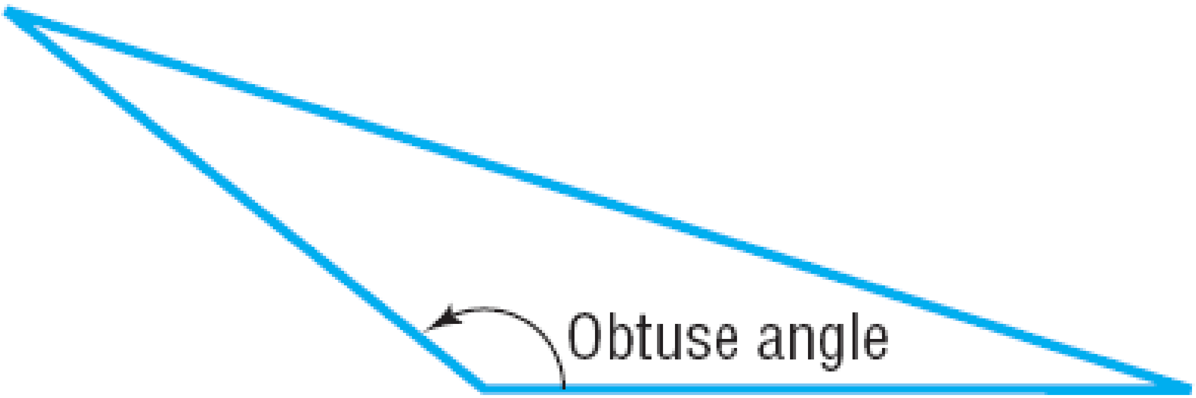
$$\cos\alpha + \cos\beta = 2\cos\frac{\alpha + \beta}{2}\cos\frac{\alpha - \beta}{2}$$

$$\cos\alpha - \cos\beta = -2\sin\frac{\alpha + \beta}{2}\sin\frac{\alpha - \beta}{2}$$

Section 7.1 Law of Sines



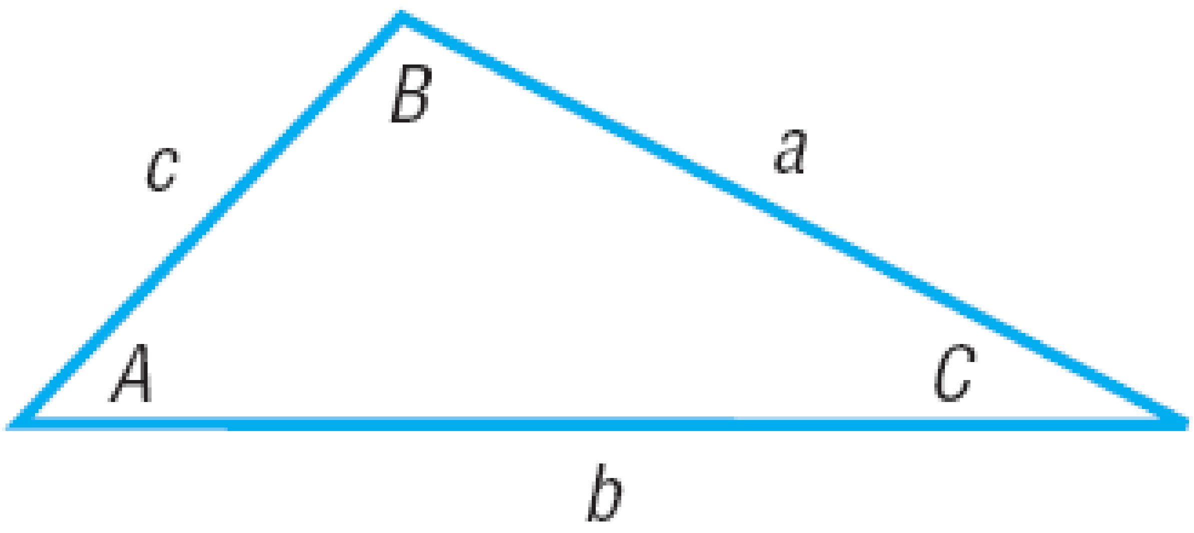
(a) All angles are acute



(b) Two acute angles and one obtuse angle

Oblique Triangle

(None of the angles is a right angle)

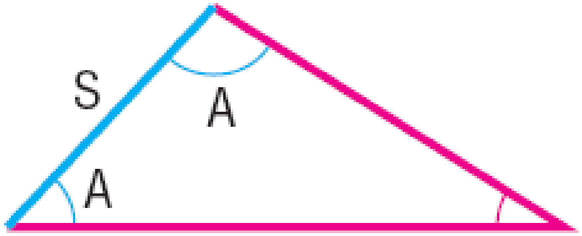


Case 1: One side and two angles are known (ASA or SAA).

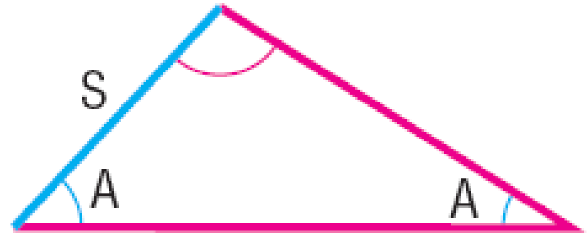
Case 2: Two sides and the angle opposite one of them are known (SSA).

Case 3: Two sides and the included angle are known (SAS).

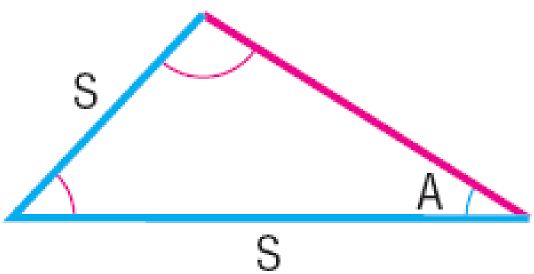
Case 4: Three sides are known (SSS).



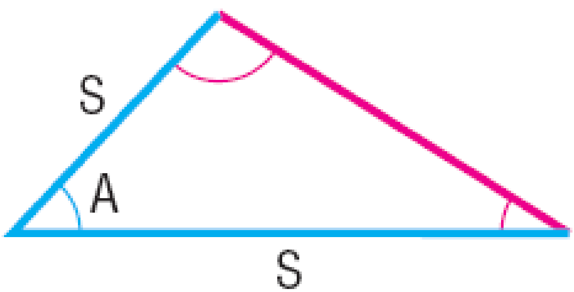
Case 1: ASA



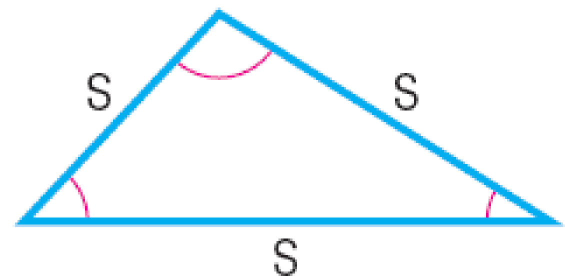
Case 1: SAA



Case 2: SSA



Case 3: SAS



Case 4: SSS

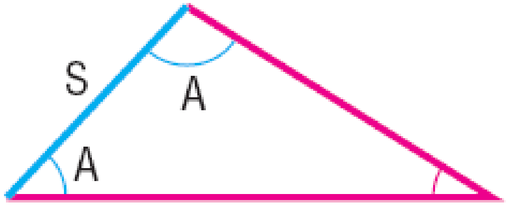
The **Law of Sines** is used to solve triangles for which Case 1 or 2 holds.

THEOREM

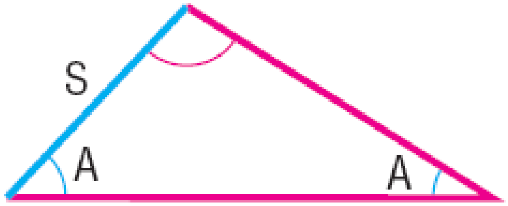
Law of Sines

For a triangle with sides a, b, c and opposite angles A, B, C, respectively,

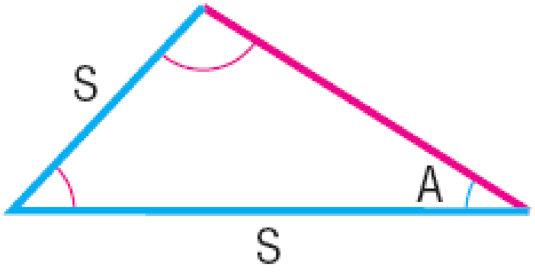
$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \qquad \textbf{(1)}$$



Case 1: ASA



Case 1: SAA



Case 2: SSA

$$\frac{\sin A}{a} = \frac{\sin B}{b} \qquad \frac{\sin A}{a} = \frac{\sin C}{c} \qquad \frac{\sin B}{b} = \frac{\sin C}{c}$$

$$A + B + C = 180^\circ$$

1 Solve SAA or ASA Triangles

EXAMPLE Using the Law of Sines to Solve an SAA Triangle

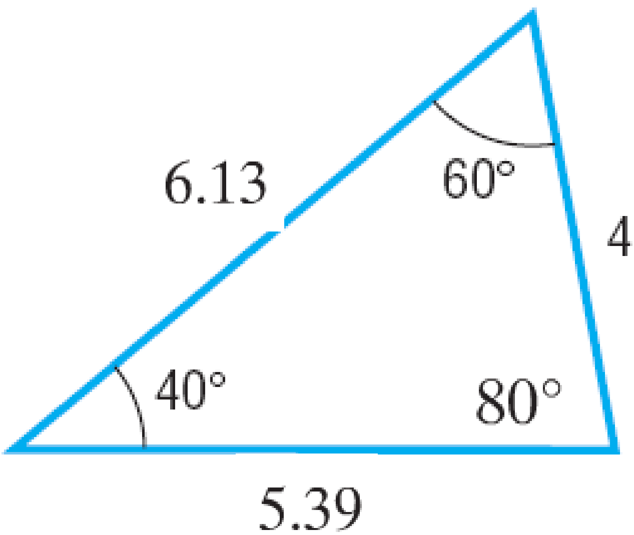
Solve the triangle: $A = 40°, B = 60°, a = 4$

$40° + 60° + C = 180° \qquad C = 80°$

$$\frac{\sin A}{a} = \frac{\sin B}{b} \qquad \frac{\sin A}{a} = \frac{\sin C}{c}$$

Because $a = 4$, $A = 40°$, $B = 60°$, and $C = 80°$, we have

$$\frac{\sin 40°}{4} = \frac{\sin 60°}{b} \qquad \frac{\sin 40°}{4} = \frac{\sin 80°}{c}$$



$$b = \frac{4 \sin 60°}{\sin 40°} \approx 5.39$$

$$c = \frac{4 \sin 80°}{\sin 40°} \approx 6.13$$

EXAMPLE Using the Law of Sines to Solve an ASA Triangle

Solve the triangle: $A = 35°, B = 15°, c = 5$

$35° + 15° + C = 180°$ $\quad C = 130°$

$$\frac{\sin A}{a} = \frac{\sin C}{c}$$

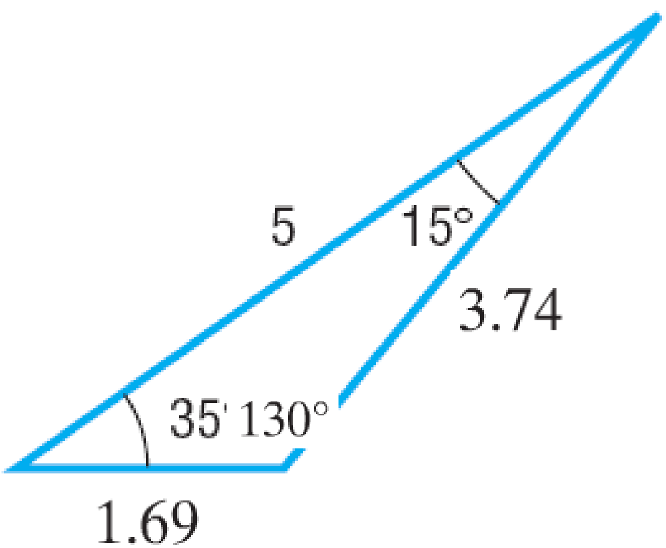
$$\frac{\sin 35°}{a} = \frac{\sin 130°}{5}$$

$$a = \frac{5 \sin 35°}{\sin 130°} \approx 3.74$$

$$\frac{\sin B}{b} = \frac{\sin C}{c}$$

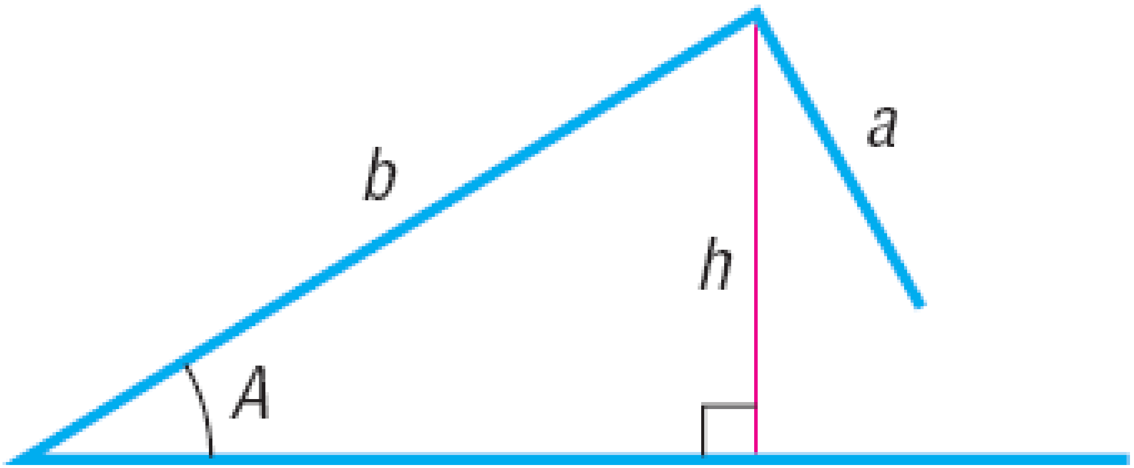
$$\frac{\sin 15°}{b} = \frac{\sin 130°}{5}$$

$$b = \frac{5 \sin 15°}{\sin 130°} \approx 1.69$$



2 Solve SSA Triangles

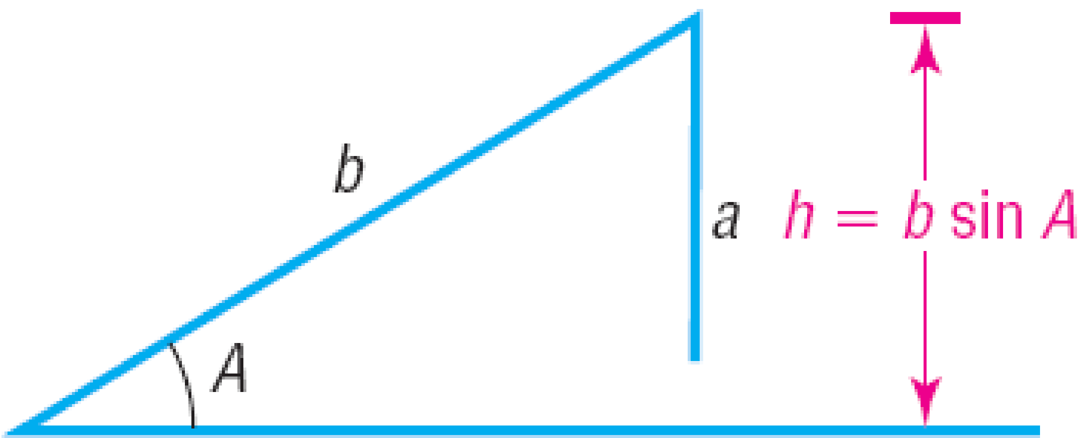
SSA --- The Ambiguous Case



$$\sin A = \frac{h}{b}$$

No Triangle If $a < h = b \sin A$, then side a is not sufficiently long to form a triangle. See Figure 14.

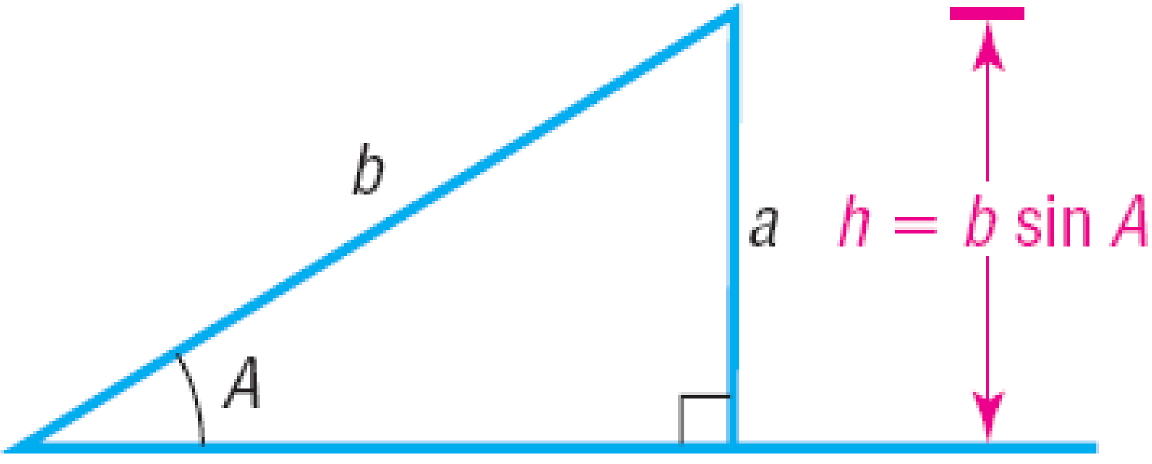
Figure 14
$a < h = b \sin A$



One Right Triangle If $a = h = b \sin A$, then side a is just long enough to form a right triangle. See Figure 15.

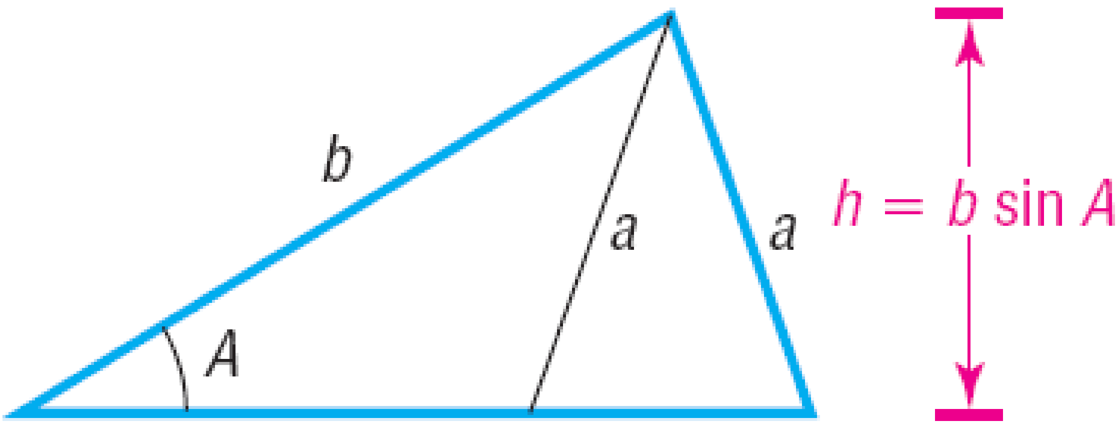
Figure 15

$a = h = b \sin A$



Two Triangles If $h = b \sin A < a$, and $a < b$ two distinct triangles can be formed from the given information. See Figure 16.

Figure 16
$b \sin A < a$ and $a < b$



One Triangle If $a \geq b$, only one triangle can be formed. See Figure 17.

Figure 17
$a \geq b$

EXAMPLE

Using the Law of Sines to Solve an SSA Triangle (One Solution)

Solve the triangle: $a = 3, b = 2, A = 40°$

$$\frac{\sin 40°}{3} = \frac{\sin B}{2} \qquad \sin B = \frac{2 \sin 40°}{3} \approx 0.43$$

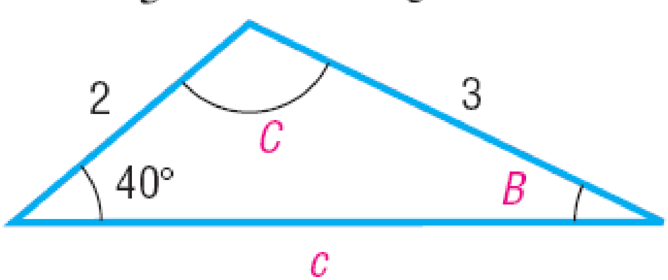
There are two angles B, $0° < B < 180°$, for which $\sin B \approx 0.43$.

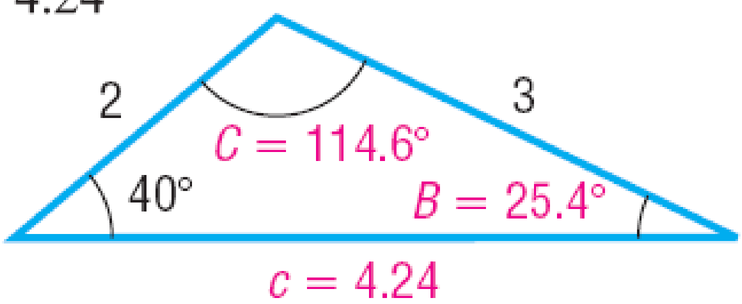
$$B_1 \approx 25.4° \quad \text{and} \quad B_2 \approx 180° - 25.4° = 154.6°$$

The second possibility, $B_2 \approx 154.6°$, is ruled out, because $A = 40°$ makes $A + B_2 \approx 194.6° > 180°$. Now, using $B_1 \approx 25.4°$, we find that

$$C = 180° - A - B_1 \approx 180° - 40° - 25.4° = 114.6°$$

$$\frac{\sin 40°}{3} = \frac{\sin 114.6°}{c} \qquad c = \frac{3 \sin 114.6°}{\sin 40°} \approx 4.24$$





EXAMPLE

Using the Law of Sines to Solve an SSA Triangle (Two Solutions)

Solve the triangle: $a = 6, b = 8, A = 35°$

$$\frac{\sin 35°}{6} = \frac{\sin B}{8}$$

$$\sin B = \frac{8 \sin 35°}{6} \approx 0.76$$

$$B_1 \approx 49.9°$$

$$B_2 \approx 180° - 49.9° = 130.1°$$

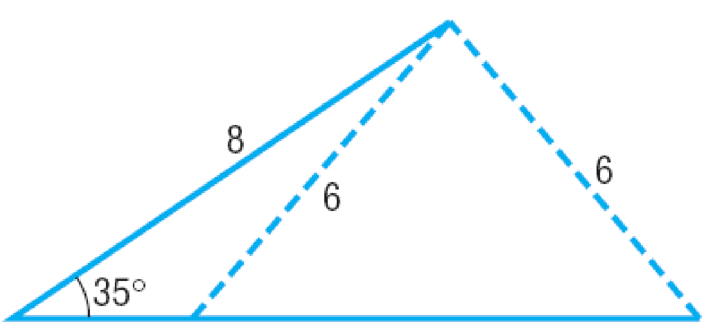
$$C_1 = 180° - A - B_1 \approx 95.1°$$

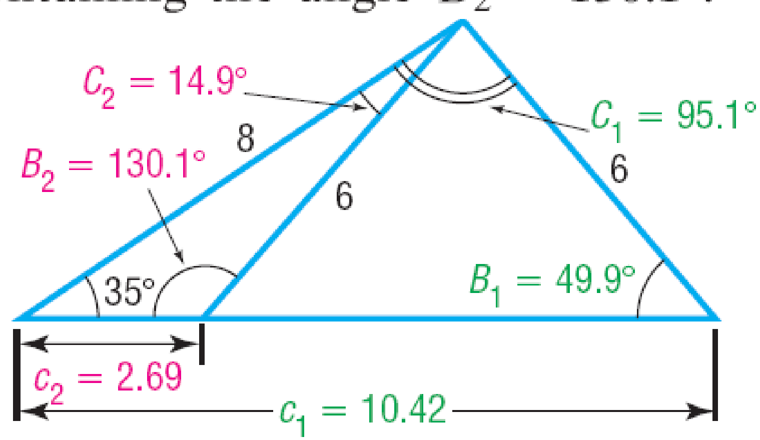
$$\frac{\sin 35°}{6} = \frac{\sin 95.1°}{c_1} \qquad c_1 = \frac{6 \sin 95.1°}{\sin 35°} \approx 10.42$$

$$C_2 = 180° - A - B_2 \approx 14.9°$$

$$\frac{\sin 35°}{6} = \frac{\sin 14.9°}{c_2} \qquad c_2 = \frac{6 \sin 14.9°}{\sin 35°} \approx 2.69$$

For both choices of B, we have $A + B < 180°$. There are two triangles, one containing the angle $B_1 \approx 49.9°$ and the other containing the angle $B_2 \approx 130.1°$.





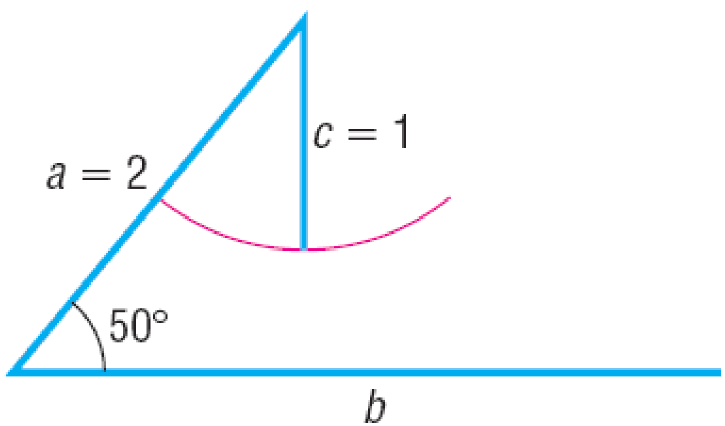
EXAMPLE

Using the Law of Sines to Solve an SSA Triangle (No Solution)

Solve the triangle: $a = 2, c = 1, C = 50^\circ$

$$\frac{\sin A}{2} = \frac{\sin 50^\circ}{1} \qquad \sin A = 2 \sin 50^\circ \approx 1.53$$

Since there is no angle A for which $\sin A > 1$, there can be no triangle with the given measurements. Figure 20 illustrates the measurements given. Notice that, no matter how we attempt to position side c, it will never touch side b to form a triangle.



EXAMPLE Rescue at Sea

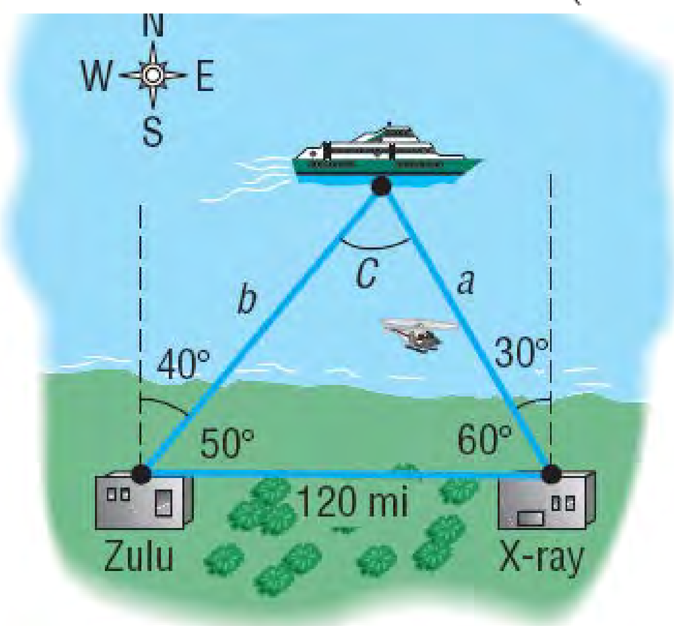
(a) How far is each station from the ship?

(b) If a helicopter capable of flying 200 miles per hour is dispatched from the nearest station to the ship, how long will it take to reach the ship?

Station Zulu is about 111 miles from the ship, and Station X-ray is about 98 miles from the ship.

(b) The time t needed for the helicopter to reach the ship from Station X-ray is found by using the formula

$$(\text{Rate}, r)(\text{Time}, t) = \text{Distance}, a$$

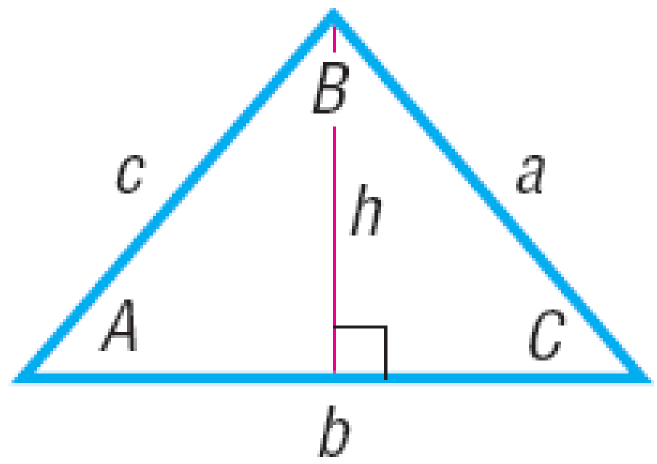


$$t = \frac{a}{r} = \frac{97.82}{200} \approx 0.49 \text{ hour} \approx 29 \text{ minutes}$$

It will take about 29 minutes for the helicopter to reach the ship.

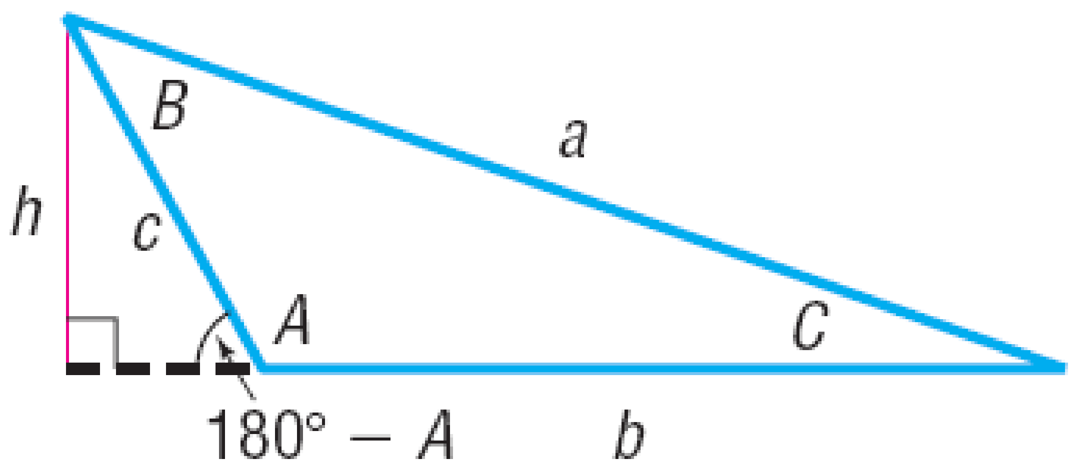
Proof of the Law of Sines

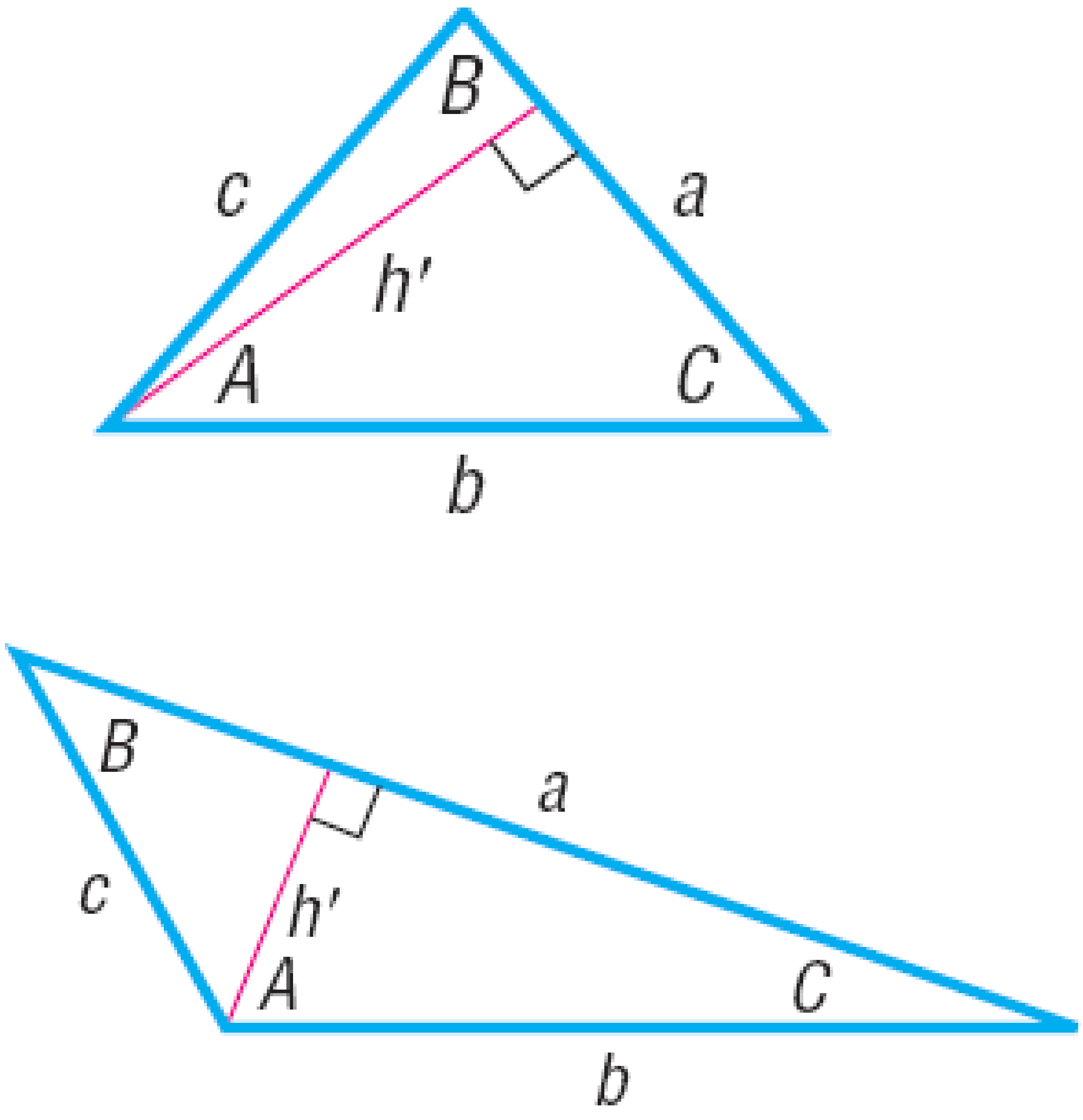
$$\sin C = \frac{h}{a}$$



$$h = c \sin A$$

$$h = a \sin C$$




B
c
a
h'
A
C
b
B
a
c
h'
A
C
b