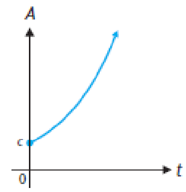
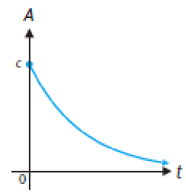
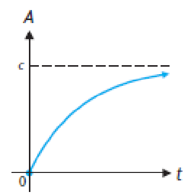
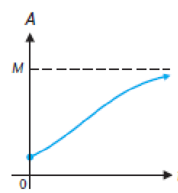
Section 4.2 Exponential Models

Unlimited growth	$A = A_0 e^{kt}$ $k > 0$		Short-term population growth (people, bacteria, etc.); growth of money at continuous compound interest
Exponential decay	$A = A_0 e^{-kt}$ $k > 0$		Radioactive decay; light absorption in water, glass, and the like; atmospheric pressure; electric circuits
Limited growth	$A = c(1 - e^{-kt})$ $c, k > 0$		Learning skills; sales fads; company growth; electric circuits
Logistic growth	$A = \frac{M}{1 + ce^{-kt}}$ $c, k, M > 0$		Long-term population growth; epidemics; sales of new products; spread of rumors; company growth

Population growth

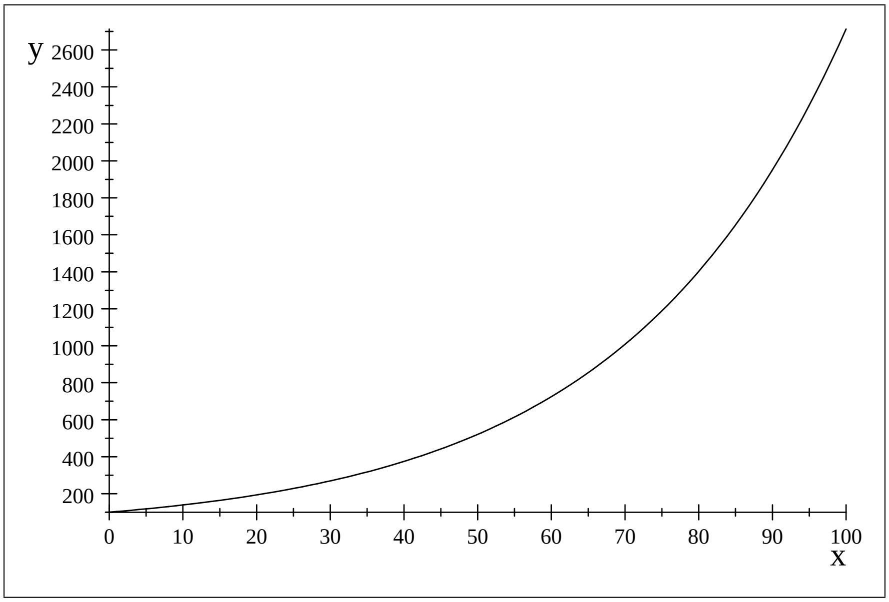
Mexico has a population of around 100 million people, and it is estimated that the population will double in 21 years. If the population continues to grow at the same rate. What will the population be in 15 years?

$$\begin{aligned} A &= A_0 e^{kt} \\ A_0 &= 100 \\ 200 &= 100e^{21k} \\ e^{21k} &= 2 \Rightarrow 21k = \ln 2 \\ &\Rightarrow k = \frac{\ln 2}{21} = 3.3007 \times 10^{-2} \end{aligned}$$

In 15 years,

$$\begin{aligned} A &= 100e^{15k} \\ &= 100e^{15\left(\frac{\ln 2}{21}\right)} \\ &\approx 164.07 \end{aligned}$$

Here is the graph of the growth of the population $A(t) = 100e^{\left(\frac{\ln 2}{21}\right)t}$



Radioactive decay

The radioactive isotope gallium 67 used in the diagnosis of malignant tumors has a biological half-life of 46.5 hours. If we start with 100 milligrams of the isotope, how many milligrams will be left after 1 week

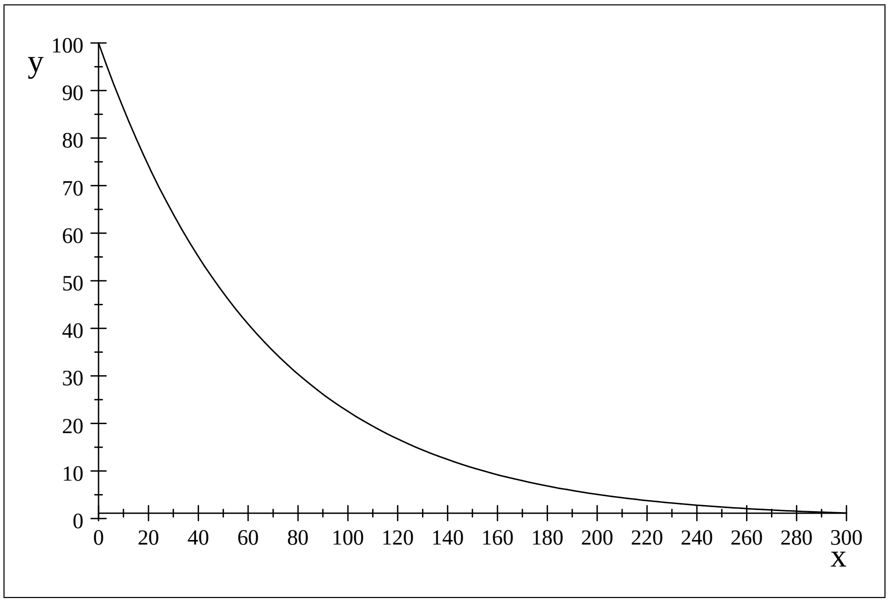
$$\begin{aligned} A(t) &= A_0 e^{-kt} \\ A(t) &= 100e^{-kt} \end{aligned}$$

$$\begin{aligned} 50 &= 100e^{-46.5k} \\ A(t) &= A_0 e^{-kt} e^{-46.5k} = 0.5 \\ -46.5k &= \ln(0.5) \\ k &= -\frac{\ln(0.5)}{46.5} \end{aligned}$$

In a week, we have $(24)\,7 = 168$ hours, thus,

$$\begin{aligned} A(168) &= 100e^{\frac{\ln(0.5)}{46.5}168} \\ &= 8.1735 \end{aligned}$$

Here is the graph of $A(t) = 100e^{\left(\frac{\ln(0.5)}{46.5}\right)t}$



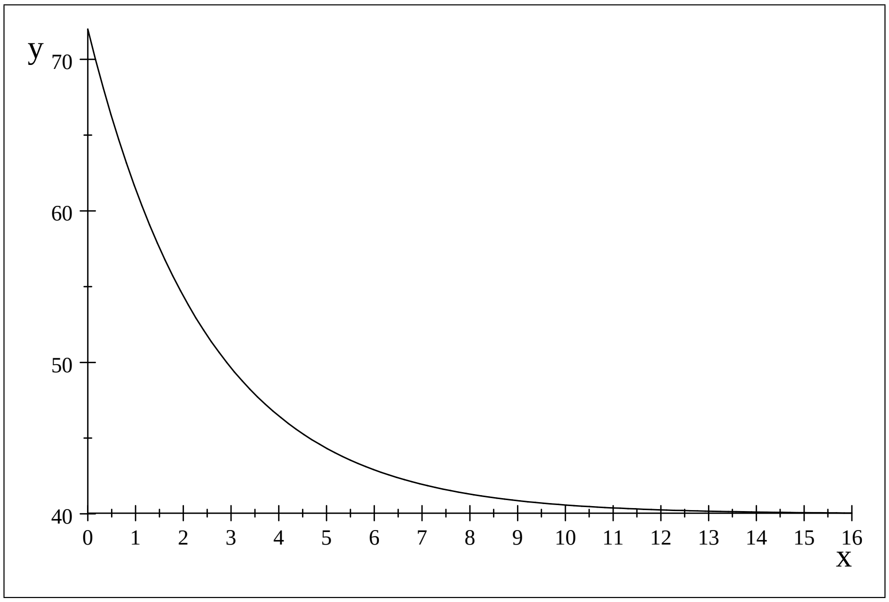
Newton's Law of cooling

This law states that the rate at which an object cools is proportional to the difference in temperature between the object and its surrounding medium. The temperature T of an object t hours later is given by

$$T = T_m + (T_0 - T_m)\, e^{-kt}$$

T_m is the temperature of the surrounding, T_0 is the initial temperature of the object. Suppose a bottle of wine at a room temperature of 72 degree is placed in the refrigerator to cool off. If the temperature in the refrigerator is kept at 40 degree and $k = 0.4$, find the temperature of the wine after 3 hours

$$\begin{aligned} T(t) &= 40 + (72 - 40)\, e^{-0.4t} \\ &= 32e^{-0.4t} + 40 \end{aligned}$$



Temperature T

$$\begin{aligned} T(3) &= 32e^{-0.4(3)} + 40 \\ &= 49.638 \end{aligned}$$