Section 1-6
Inverse functions
Lecture notes

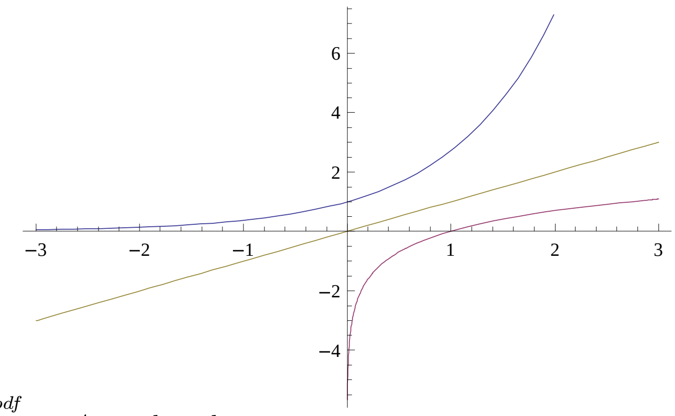
Definition 1 *Given a function f, the inverse function of f is defined as the function f^{-1} such that their compositions yield to the identity function. In other words,*

$$\left(f^{-1} \circ f\right)(x) = x$$
$$\left(f \circ f^{-1}\right)(x) = x.$$

Geometrically speaking,the graphs of inverse functions are symmetrical with respect to the diagonal line $y = x$.

and

inverses



$_1$*.pdf*

A graph and its inverse

Definition 2 *A function is one-to-one or injective, if and only if $f(x) = f(y)$ implies that $x = y$. It's a mathematical fact a function' has a defined inverse on some interval I if and only if it is one-to-one. Examples of one-to-one functions are the square root function or the cube function. However, the square function is not one-to-one.*

Example

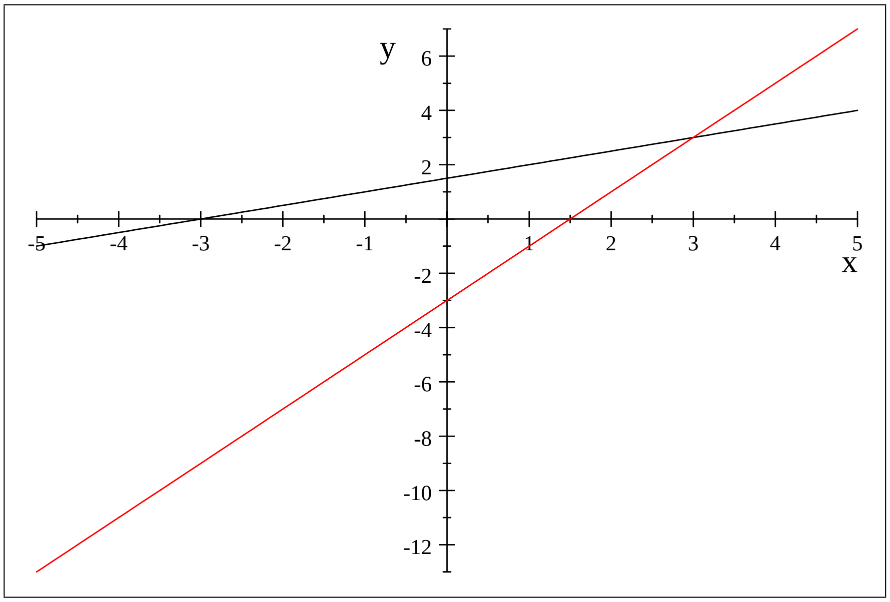
1. Decide if these 2 functions are inverses: $f(x) = 2x - 3, g(x) = \frac{x+3}{2}$
2. Find f^{-1} for $f(x) = \sqrt{x-3}$
3. Find f^{-1} for $f(x) = \frac{5-3x}{7-4x}$
4. Find f^{-1} for $f(x) = 3 - \sqrt{5-x}$

Solution

1.

$$\begin{aligned}(f \circ g)(x) &= f\left(\frac{x+3}{2}\right) \\ &= 2\left(\frac{x+3}{2}\right) - 3 \\ &= x\end{aligned}$$

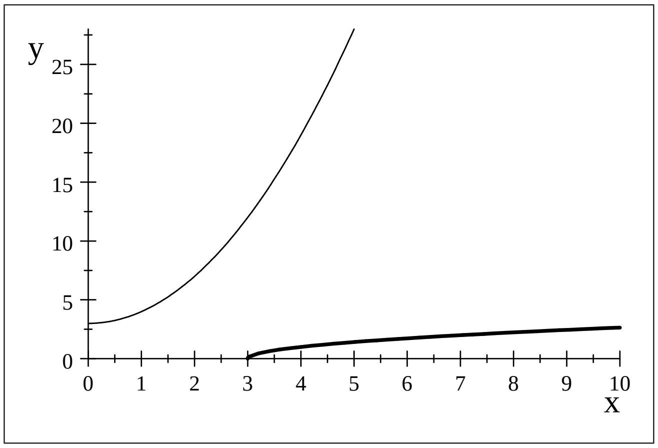
$$\begin{aligned}(g \circ f)(x) &= g(2x-3) \\ &= \frac{(2x-3)+3}{2} \\ &= \frac{2x}{2} \\ &= x\end{aligned}$$



2. Finding f^{-1} for $f(x) = \sqrt{x-3}$. Put $y = \sqrt{x-3}$ we solve for x,

$$\begin{aligned}y &= \sqrt{x-3} \\ y^2 &= x-3 \\ x &= y^2+3 \\ f^{-1}(x) &= x^2+3\end{aligned}$$

Check by plotting that the graphs are symmetrical with respect to the diagonal line. $x^2 + 3$



3. Find f^{-1} for $f(x) = \frac{5-3x}{7-4x}$. We put $y = \frac{5-3x}{7-4x}$ and we solve for x.

$$\begin{aligned} y &= \frac{5-3x}{7-4x} \\ 5-3x &= (7-4x)\,y \\ 5-3x &= 7y-4xy \\ -3x &= 7y-4xy-5 \\ -3x+4xy &= 7y-5 \\ x(-3+4y) &= 7y-5 \\ x &= \frac{7y-5}{-3+4y} \end{aligned}$$

Thus,

$$f^{-1}(x) = \frac{7x-5}{-3+4x}$$

We check

$$\begin{aligned} f^{-1}(f(x)) &= \frac{7\left(\frac{5-3x}{7-4x}\right)-5}{-3+4\left(\frac{5-3x}{7-4x}\right)} \\ &= \frac{\frac{x}{4x-7}}{\frac{1}{4x-7}} \\ &= \left(\frac{x}{4x-7}\right)\left(\frac{4x-7}{1}\right) \\ &= x \end{aligned}$$

Similarly, $f(f^{-1}(x)) = x$

4. Find f^{-1} for $f(x) = 3 - \sqrt{5-x}$. Put $y = 3 - \sqrt{5-x}$ and solve for x

$$\begin{aligned} y &= 3 - \sqrt{5-x} \\ y - 3 &= -\sqrt{5-x} \\ -y + 3 &= \sqrt{5-x} \\ (-y+3)^2 &= 5 - x \\ (-y+3)^2 - 5 &= -x \\ x &= 5 - (-y+3)^2 \\ &= -y^2 + 6y - 4 \end{aligned}$$

Thus, $f^{-1}(x) = -x^2 + 6x - 4$

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