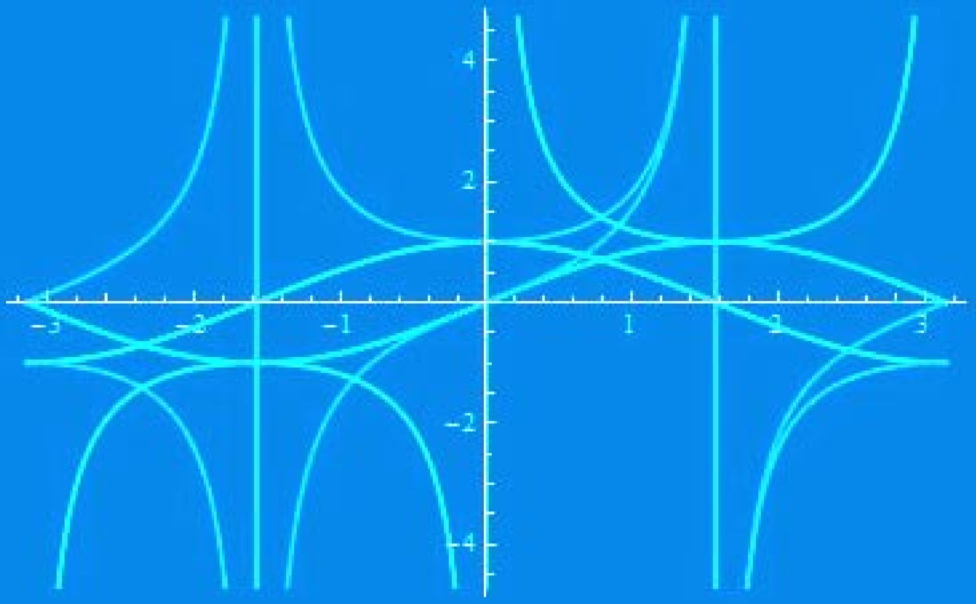
Lecture notes for Precalculus

Vignon S. Oussa



Designed as a supplemenary tool for the text book of Raymond Barnett,
Precalculus Graph and Models, Third Edition. McGraw-Hill, 2009.

Some comments

This ebook was made at the end of a precalculus and trigonometry semester-long course taught at Saint Louis University. I compiled all the lecture notes written as a supplementary tool to the textbook by Raymond Barnett, Michael Ziegler, Karl Byleen, and David Sobecki titled "Precalculus-Graph and Models," Third Edition. McGraw-Hill, 2009. I should warn the user of these notes that they do not cover the entire book but only the chapters that were taught in class. There are a few typos and a few mild mistakes in some sections of this ebook either discovered during lectures or after revision. Necessary corrections will be made on the next version and the second half portion of the ebook which is presently handwritten, will be typed up in Latex.

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Section 1.2
Functions

By definition, a **function** is a relation in which every element in the domain corresponds to one and only one element in the range. Alternatively, a function is a set of ordered pairs with the property that no two pairs have the same element in the first component and different element in the second component. The set of all elements in the first component is called the **domain** of the function, and set of all element in the second component is called the **range** of the function.

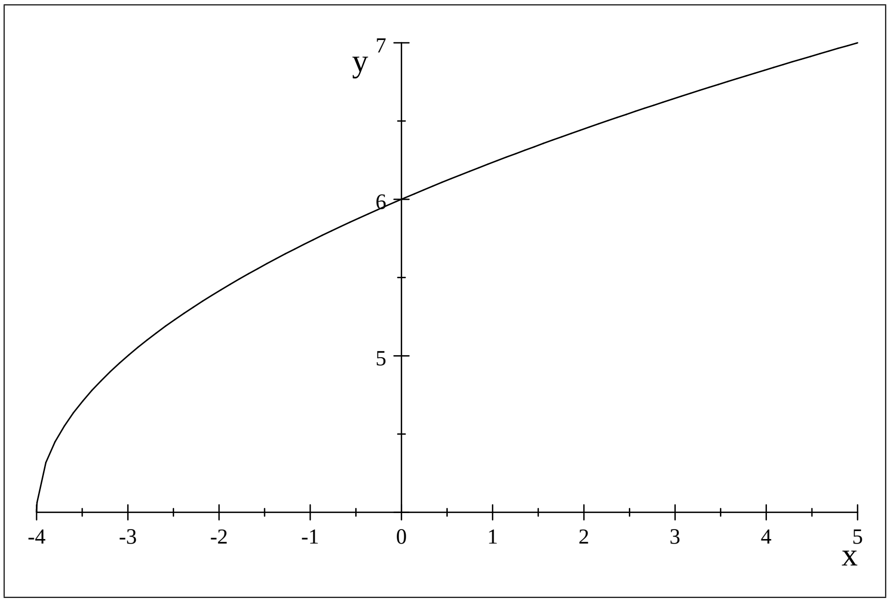
The **vertical line test**, is a test applied to the graph of a function which tell us whether the graph represents a function or not.If there exits any vertical line that meets the graph of the function at more than one point, the relation is not a function. Otherwise, it is a function.

Examples

1. Determine if the following relation is a function
$$y = 4 + \sqrt{x+4}$$
2. Find the domain of the functions: $f(x) = 4+\sqrt{x+4}, g(x) = \frac{15}{v^2-16}, h(x) = \frac{\sqrt{x-3}}{x+2}$
3. Given the function $f(x) = \frac{15}{x-2}, g(x) = 2x^2+3x-6, h(x) = \frac{1}{\sqrt{x-1}}$, compute $f(2), g(0), h(3)$
4. Given the function $f(x) = x^2 + 2x - 1$, compute $f(2), f(2+h)$, and $\frac{f(2+h)-f(2)}{h}$

Partial Solutions

1. Plotting the graph of the relation $y = 4 + \sqrt{x+4}$,we obtain



Clearly, the no vertical line meets the graph of this function at more than one point. Thus, by the vertical line test, the relation is a function.

2. Finding the domain of $f(x) = 4 + \sqrt{x+4}, g(x) = \frac{15}{v^2-16}$ Let D_f denotes the domain of $f(x)$

$$\begin{aligned} D_f &= \{x \in \mathbb{R} : x + 4 \geq 0\} \\ x + 4 &\geq 0 \Longrightarrow x \geq -4 \\ &\Rightarrow x \in [-4, \infty) \end{aligned}$$

Let D_g denotes the domain of the function $g(x)$

$$\begin{aligned} D_g &= \{v \in \mathbb{R} : v^2 - 16 \neq 0\} \\ v^2 - 16 &= 0 \Rightarrow (v-4)(v+4) = 0 \\ &\Rightarrow v - 4 = 0 \text{ or } v + 4 = 0 \\ &\Rightarrow v = 4 \text{ or } v = -4 \end{aligned}$$

Thus,

$$\begin{aligned} D_g &= \{v \in \mathbb{R} : v^2 - 16 \neq 0\} \\ &= \{v \in \mathbb{R} : v^2 - 16 \neq 0\} \\ &= \{v \in \mathbb{R} : v \neq 4 \text{ and } v \neq -4\}. \\ &= (-\infty, -4) \cup (-4, 4) \cup (4, \infty) \end{aligned}$$

3. Given the function $f(x) = \frac{15}{x-2}, g(x) = 2x^2 + 3x - 6, h(x) = \frac{1}{\sqrt{x-1}}$, we

compute $f(2), g(0), h(3)$

$$\begin{aligned}
f(2) &= \frac{15}{2-2} = \frac{15}{0} \text{ is undefined} \\
g(0) &= 2 \cdot 0^2 + 3 \cdot 0 - 6 = -6 \\
h(3) &= \frac{1}{\sqrt{3-1}} = \frac{1}{\sqrt{2}}
\end{aligned}$$

4. Given the function $f(x) = x^2+2x-1$, we compute $f(2+h)$, and $\frac{f(2+h)-f(2)}{h}$.

$$\begin{aligned}
f(2+h) &= (2+h)^2 + 2(2+h) - 1 \\
&= 4 + 4h + h^2 + 4 + 2h - 1 \\
&= h^2 + 6h + 7.
\end{aligned}$$

$$\begin{aligned}
\frac{f(2+h) - f(2)}{h} &= \frac{h^2 + 6h + 7 - \left(2^2 + 2 \cdot 2 - 1\right)}{h} \\
&= \frac{h^2 + 6h}{h} \\
&= \frac{(h+6)\,h}{h} \\
&= (h+6)\,.
\end{aligned}$$

Homework:

Page 42#1, 13, 17, 25, 31, 35, 40, 43, 52, 57, 61, 63.

Section 1-3
Functions: Graphs and Properties
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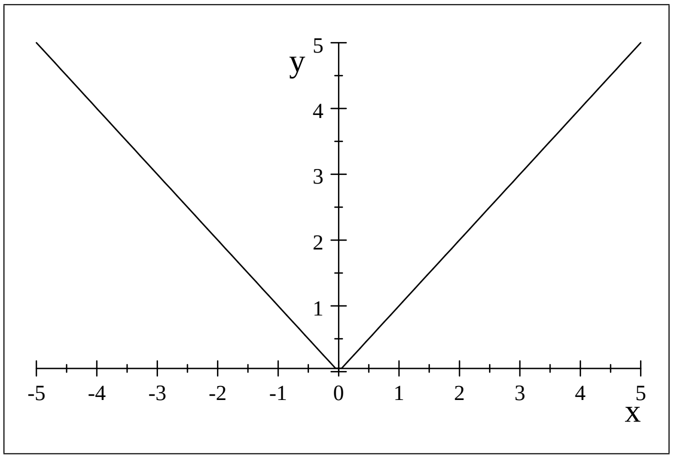
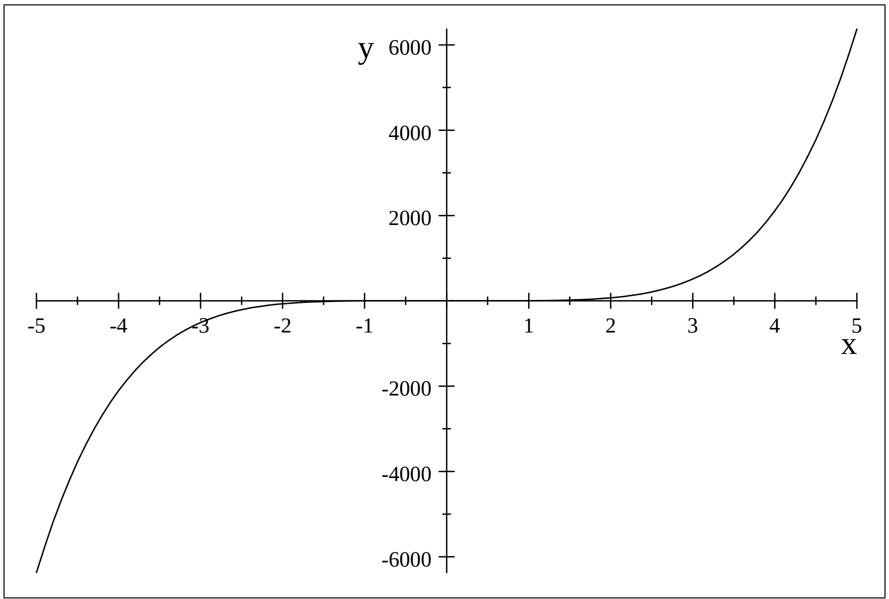
The **x-intercepts** of the graph of a function are the points (ordered pairs) at which the graph of the function meets the x-axis of the plane. The first coordinate of the x-intercepts are called the zeros of a the function.
The **y-intercepts** of the graph of a function are the points at which the graph of the function meets the y-axis.
We say a function is **increasing** on some interval I if on only if the graph of the function is rising on that interval. A function is **decreasing** on some interval I if and only if the graph of the function is falling on the interval I. The graph of a function f is constant on some interval I if the graph of the function is horizontal line on the interval I.
The value $f(c)$ is called a **local maximum** if there exits some interval I such that $f(x) \leq f(c)$ for any $x \in I$. The value $f(c)$ is called a **local minimum** if there exits some interval I such that $f(x) \geq f(c)$ for any $x \in I$.
A **piecewise** defined function is a function defined by different functions on different intervals. For example, the absolute value function $f(x) = |x|$ is piecewise defined

$$f(x) = \begin{cases} x & \text{if } x > 0 \\ -x & \text{if } x < 0 \end{cases}$$

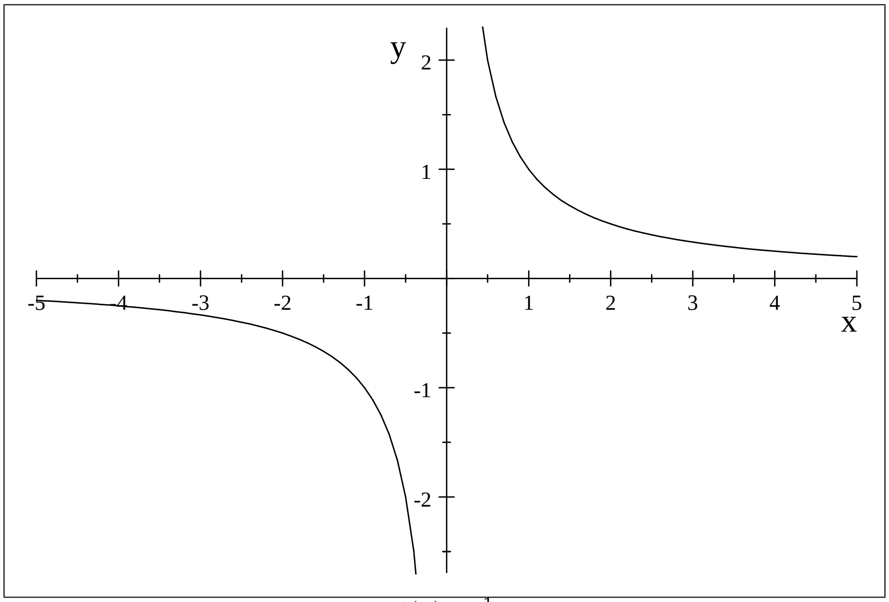


Graph of absolute value of x

A function is said to be **continuous** if its graph has no gaps, jumps or holes. For example, the function $f(x) = x^3 + 2x^5 - x + 1$ is continuous, while the function $f(x) = \frac{1}{x}$ is not continuous at zero. To see these facts, check the graphs of the functions:



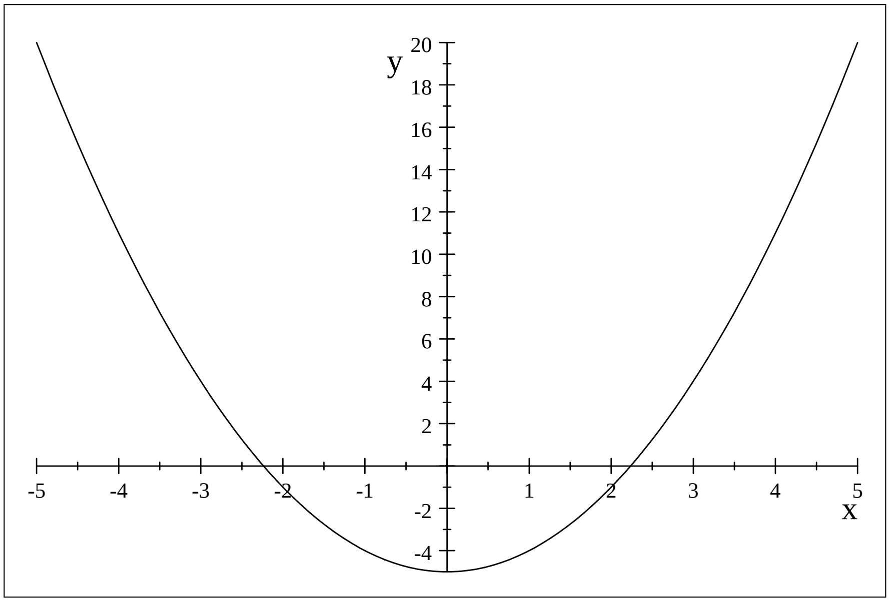
$f(x) = x^3 + 2x^5 - x + 1$



$f(x) = \frac{1}{x}$

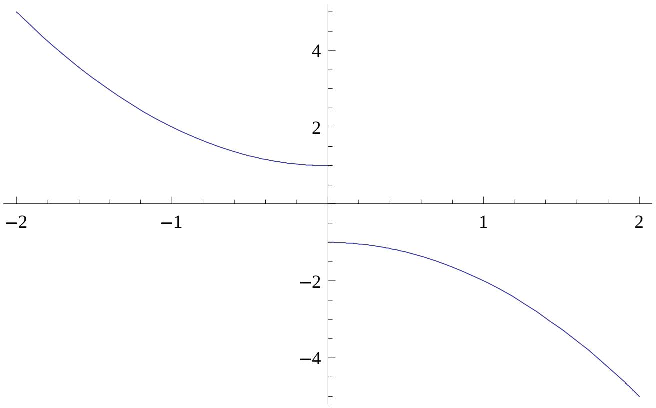
Example

1. Given the following graph, find the x, and y intercepts of the graph. Find the domain, and the range of the function, and the interval on which the function is increasing, and decreasing.



2. Plot the graph of the following piecewise defined function:

$$f(x) = \begin{cases} x^2 + 1 & \text{if } x < 0 \\ -x^2 - 1 & \text{if } x > 0 \end{cases}$$



Homework:

Page 63 $\#6, 13, 19, 22, 27, 29, 34, , 3741, 41, 45, 67, 70$

Math 141-01
Section 1.4
Functions: Graphs and Transformations

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Library functions:

- $f(x) = x$
- $f(x) = |x|$
- $f(x) = x^2$
- $f(x) = x^3$
- $f(x) = \sqrt{x}$
- $f(x) = \sqrt[3]{x}$

Vertical shifting

Given a function $f(x)$, the graph of $g(x) = f(x) + a$, is obtained by shifting vertically the graph of f, a units. If a is positive, the shift is upward and if a is negative, the shift is downward.

Horizontal shifts

Given a a positive number, and a function $f(x)$, the graph of the function $g(x) = f(x - a)$ is obtained by shifting horizontally the graph of f a units to the right, and the graph of the function $g(x) = f(x + a)$ is obtained by shifting horizontally the graph of f a units to the left.

Reflection

Given a function $f(x)$, the graph of $-f(x)$ is obtained by reflecting the graph of f with respect to the x-axis. The graph of $f(-x)$ is obtained by reflecting the graph of f with respect to the y-axis.

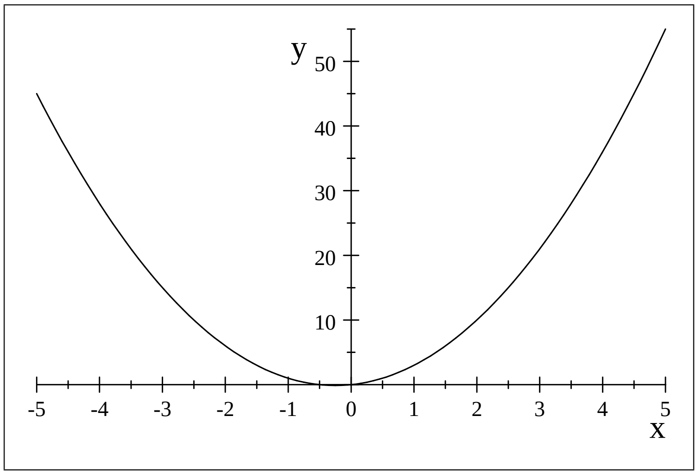
Even and odd

A function f is even if and only if $f(-x) = f(x)$. f is odd if and only if $f(-x) = -f(x)$. The graph of an even function is symmetrical with respect to the y-axis, and the graph of an odd function is symmetrical with respect to the x-axis.

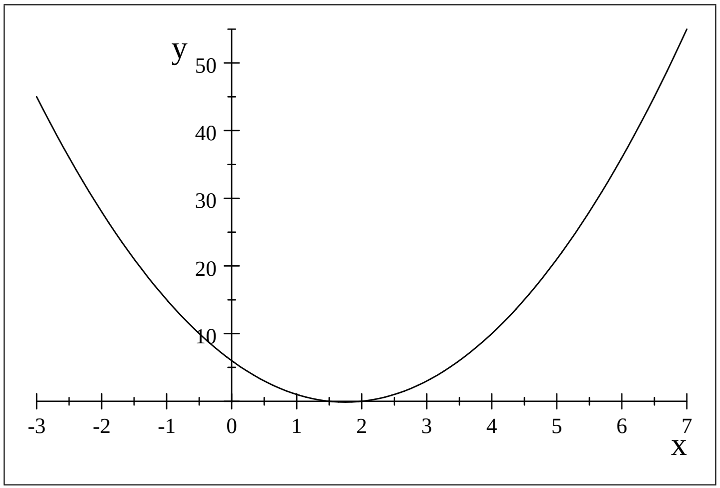
Examples:

1. Given $f(x) = 2x^2 + x$, plot the graph of the following functions $f(x) - 2, f(x) + 2$, $f(x - 2), f(x + 2), -f(x), f(-x), 2f(x)$,and $f(2x)$
2. By using graph transformations, plot the following function using a library function: $f(x) = -\sqrt{x-2} + 2$
3. Determine if the functions defined by $f(x) = x^3 + x^2, x^5, x^8 - x^4$ are either even, odd or neither.

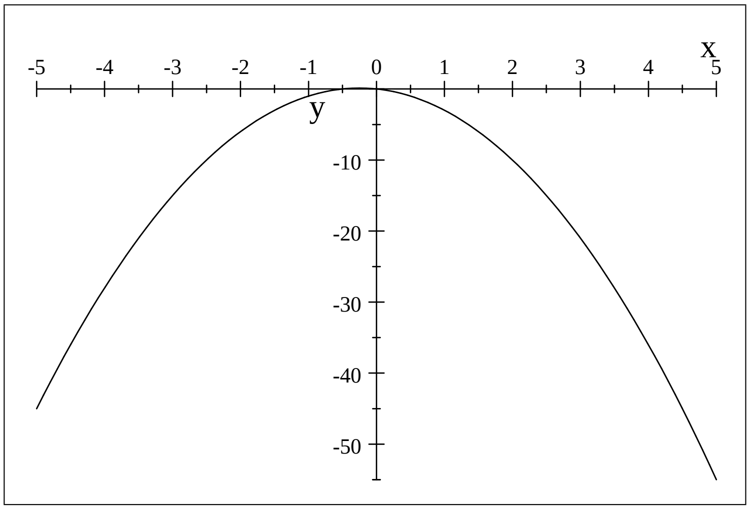
Partial solutions:

1. 

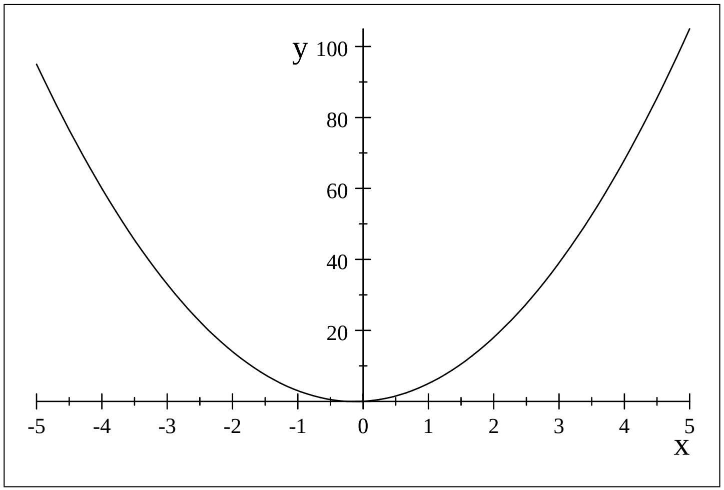
$f(x) = 2x^2 + x$



$f(x - 2)$

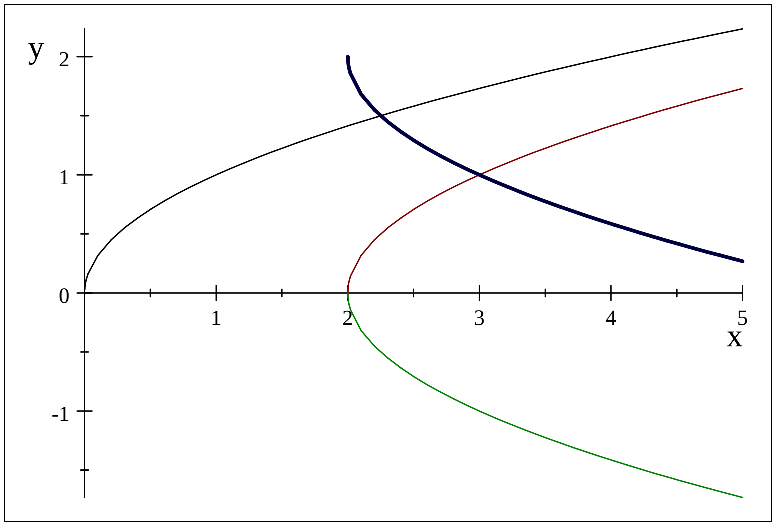


$f(-x)$



$f(2x)$

2. By shifting the graph of $\sqrt{x}$ two unit to the right, then we reflect with respect to the y-axis and shift vertically 3 units $-\sqrt{x-2}+2, \sqrt{x}$



3. Determining if the functions defined by $f(x) = x^3 + x^2, x^5, x^8 - x^4$ are either even, odd or neither.$f(x) = x^3 + x^2$ is neither because

$$\begin{aligned} f(-x) &= (-x)^3 + (-x)^2 \\ &= -x^3 + x^2 \neq f(x) \\ &\neq -f(x) = -x^3 - x^2 \end{aligned}$$

$f(x) = x^5$ is odd because,

$$\begin{aligned} f(-x) &= (-x)^5 \\ &= -x^5 \\ &= -f(x). \end{aligned}$$

$f(x) = x^8 - x^4$ is even since,

$$\begin{aligned} f(-x) &= (-x)^8 - \left(-x^4\right) \\ &= x^8 - x^4 \\ &= f(x) \end{aligned}$$

Homework: Page $79\#5, 7, 9, 11, 15, 21, 25, 27, 29, 41, 45, 47, 52$

Section 1-5
Operations on Functions; Compositions
Lecture notes

Definition 1

$$\begin{aligned}(f \pm g)(x) &= f(x) \pm g(x)\\ (fg)(x) &= f(x)g(x)\\ \left(\frac{f}{g}\right)(x) &= \frac{f(x)}{g(x)}\\ (f \circ g)(x) &= f(g(x)).\end{aligned}$$

Examples

1. Given $f(x) = \frac{1}{x-2}$ and $g(x) = \frac{x-5}{x}$, compute $\frac{f}{g}$
2. Given $f(x) = x^2 + x, g(x) = 3 - x$, compute $(f \circ g)(x)$, and $(g \circ f)(x)$
3. Find $(f \circ g)(x)$,and $(g \circ f)(x)$, if $f(x) = \sqrt{2 - x^2}, g(x) = \sqrt{1 - x}$.
4. Express $\sqrt{2 + 5x^{11}}$ as a composition of 2 functions.

Solutions

1.

$$\begin{aligned}\left(\frac{f}{g}\right)(x) &= \frac{\frac{1}{x-2}}{\frac{x-5}{x}}\\ &= \left(\frac{1}{x-2}\right)\left(\frac{x}{x-5}\right)\\ &= \frac{x}{(x-2)(x-5)}\\ &= \frac{x}{x^2 - 7x + 10}\end{aligned}$$

2.

$$\begin{aligned}(f \circ g)(x) &= f(g(x))\\ &= f(3 - x)\\ &= (3 - x)^2 + (3 - x)\\ &= x^2 - 6x + 9 + 3 - x\\ &= x^2 - 7x + 12\end{aligned}$$

$$\begin{aligned}(g \circ f)(x) &= g(f(x))\\ &= g\left(x^2 + x\right)\\ &= 3 - \left(x^2 + x\right)\\ &= -x^2 - x + 3\end{aligned}$$

3.

$$\begin{aligned}(f\circ g)(x) &= f(g(x))\\ &= f\left(\sqrt{1-x}\right)\\ &= \sqrt{2-(1-x)^2}\\ &= \sqrt{-x^2+2x+1}\end{aligned}$$

d

$$\begin{aligned}(g\circ f)(x) &= g(f(x))\\ &= g\left(\sqrt{2-x^2}\right)\\ &= \sqrt{1-\sqrt{2-x^2}}\end{aligned}$$

4. Put $f(x)=2+5x^{11}$, and $g(x)=\sqrt{x}$

$$\begin{aligned}(g\circ f)(x) &= g(2+5x^{11})\\ &= \sqrt{2+5x^{11}}\end{aligned}$$

Homework:
Page 96, $7, 8, 9, 10, 21, 23, 26, 27, 33, 36, 45, 47, 53, 65$

Section 1-6

Inverse functions

Lecture notes

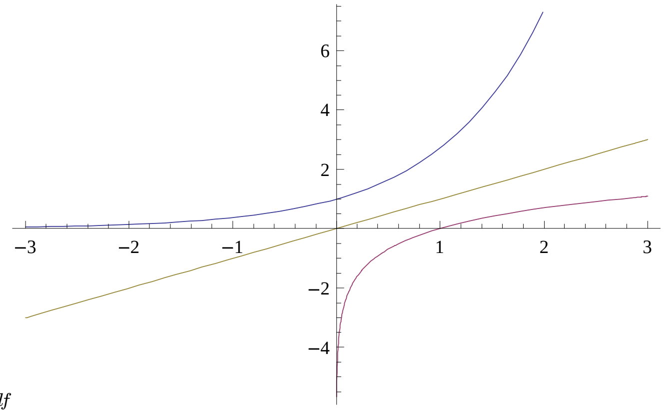
Definition 1 *Given a function f, the inverse function of f is defined as the function f^{-1} such that their compositions yield to the identity function. In other words,*

$$\begin{aligned}\left(f^{-1} \circ f\right)(x) &= x \\ \left(f \circ f^{-1}\right)(x) &= x.\end{aligned}$$

Geometrically speaking,the graphs of inverse functions are symmetrical with respect to the diagonal line $y = x$.

and

inverses



$_1$*.pdf*

A graph and its inverse

Definition 2 *A function is one-to-one or injective, if and only if $f(x) = f(y)$ implies that $x = y$. It's a mathematical fact a function' has a defined inverse on some interval I if and only if it is one-to-one. Examples of one-to-one functions are the square root function or the cube function. However, the square function is not one-to-one.*

Example

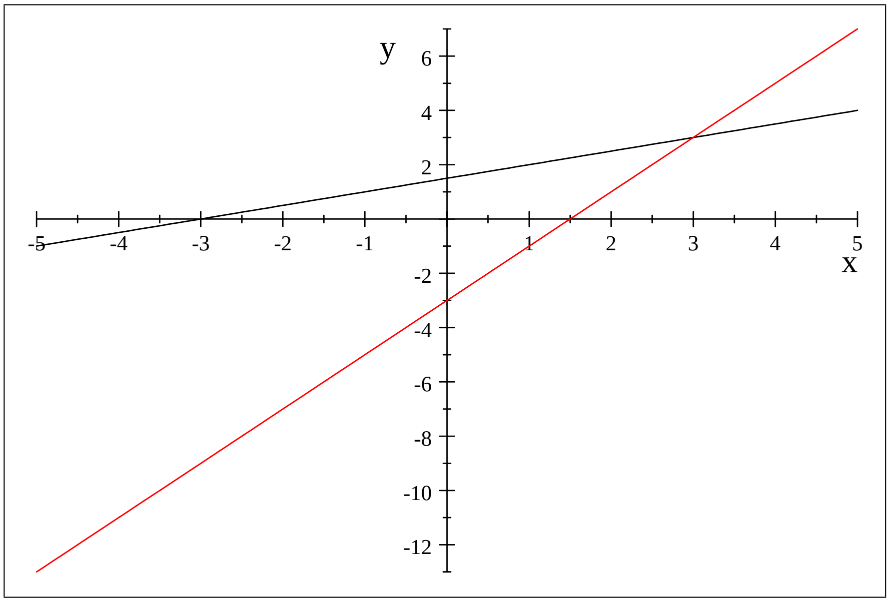
1. Decide if these 2 functions are inverses: $f(x) = 2x - 3, g(x) = \frac{x+3}{2}$
2. Find f^{-1} for $f(x) = \sqrt{x-3}$
3. Find f^{-1} for $f(x) = \frac{5-3x}{7-4x}$
4. Find f^{-1} for $f(x) = 3 - \sqrt{5-x}$

Solution

1.

$$\begin{aligned}(f\circ g)(x) &= f\left(\frac{x+3}{2}\right)\\ &= 2\left(\frac{x+3}{2}\right)-3\\ &= x\end{aligned}$$

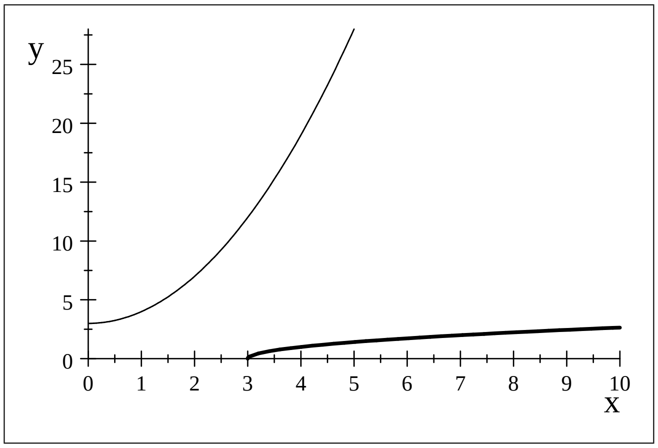
$$\begin{aligned}(g\circ f)(x) &= g(2x-3)\\ &= \frac{(2x-3)+3}{2}\\ &= \frac{2x}{2}\\ &= x\end{aligned}$$



2. Finding f^{-1} for $f(x)=\sqrt{x-3}$. Put $y=\sqrt{x-3}$ we solve for x,

$$\begin{aligned}y &= \sqrt{x-3}\\ y^2 &= x-3\\ x &= y^2+3\\ f^{-1}(x) &= x^2+3\end{aligned}$$

Check by plotting that the graphs are symmetrical with respect to the diagonal line. x^2+3



3. Find f^{-1} for $f(x) = \frac{5-3x}{7-4x}$. We put $y = \frac{5-3x}{7-4x}$ and we solve for x.

$$\begin{aligned} y &= \frac{5-3x}{7-4x} \\ 5-3x &= (7-4x)\,y \\ 5-3x &= 7y-4xy \\ -3x &= 7y-4xy-5 \\ -3x+4xy &= 7y-5 \\ x(-3+4y) &= 7y-5 \\ x &= \frac{7y-5}{-3+4y} \end{aligned}$$

Thus,

$$f^{-1}(x) = \frac{7x-5}{-3+4x}$$

We check

$$\begin{aligned} f^{-1}(f\,(x)) &= \frac{7\left(\frac{5-3x}{7-4x}\right)-5}{-3+4\left(\frac{5-3x}{7-4x}\right)} \\ &= \frac{\frac{x}{4x-7}}{\frac{1}{4x-7}} \\ &= \left(\frac{x}{4x-7}\right)\left(\frac{4x-7}{1}\right) \\ &= x \end{aligned}$$

Similarly, $f(f^{-1}\,(x)) = x$

4. Find f^{-1} for $f(x) = 3 - \sqrt{5-x}$. Put $y = 3 - \sqrt{5-x}$ and solve for x

$$\begin{aligned}
y &= 3 - \sqrt{5-x} \\
y - 3 &= -\sqrt{5-x} \\
-y + 3 &= \sqrt{5-x} \\
(-y+3)^2 &= 5 - x \\
(-y+3)^2 - 5 &= -x \\
x &= 5 - (-y+3)^2 \\
&= -y^2 + 6y - 4
\end{aligned}$$

Thus, $f^{-1}(x) = -x^2 + 6x - 4$

Homework Page 116#26, 31, 33, 35, 37, 41, 43, 47, 50, 55, 61, 65, 67

Section 2.1
Linear functions
Lecture notes

Definition 1 *A* ***linear function*** *is a function of the type*

$$f(x) = mx + b, m \neq 0, \text{ where } m, b \in \mathbb{R}.$$

m is called the slope of the line. When $m = 0$, we have a horizontal line. When $m > 0$, the line is ascending, and when $m < 0$, the line is descending.

Definition 2 *The equation of a line is written in* ***standard form*** *in the following ways:*

$$Ax + By = C.$$

Where, A, B, C are real numbers.

Definition 3 *The* ***slope-intercept*** *form equation of the line with y-intercept $(0, b)$, and slope m is computed as follows*

$$y = mx + b.$$

Definition 4 *The* ***point-slope*** *form equation of the line passing through the point (x_1, y_1) and slope m is computed as follows:*

$$y - y_1 = m(x - x_1).$$

Definition 5 *Given a line passing through 2 points $(x_1, y_1), (x_2, y_2)$. The slope of the line passing through the two points is computed as follows:*

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

Definition 6 *Two lines are* ***parallel*** *if and only if they have the same slope and different y-intercepts*

Definition 7 *Two lines are* ***perpendicular*** *if and only if the product of their slopes is -1.*

Remark 8 *The slope of a horizontal line is zero, while a vertical line has no slope.*

Examples

1. Find the x and y intercepts of the graph of the function $f(x) = \frac{1}{3}x - 5$.
2. Find the slopes of the lines passing through $(-3, -2)$ and $(-1, 8)$ or $(-4, 2)$ and $(2, -3)$.

3. Find the equation of the line passing through the points $(4, 2)$ and $(-2, 4)$ and write it in standard form, point-slope form, and in slope-intercept form.

4. Given the line L with equation $5x - 2y = 6$, find the equation of the line passing through the point $(0, 1)$ and perpendicular to L.

5. Find the equation of the line passing through the points $(2, -4)$ and such that the line is horizontal.

Partial solutions

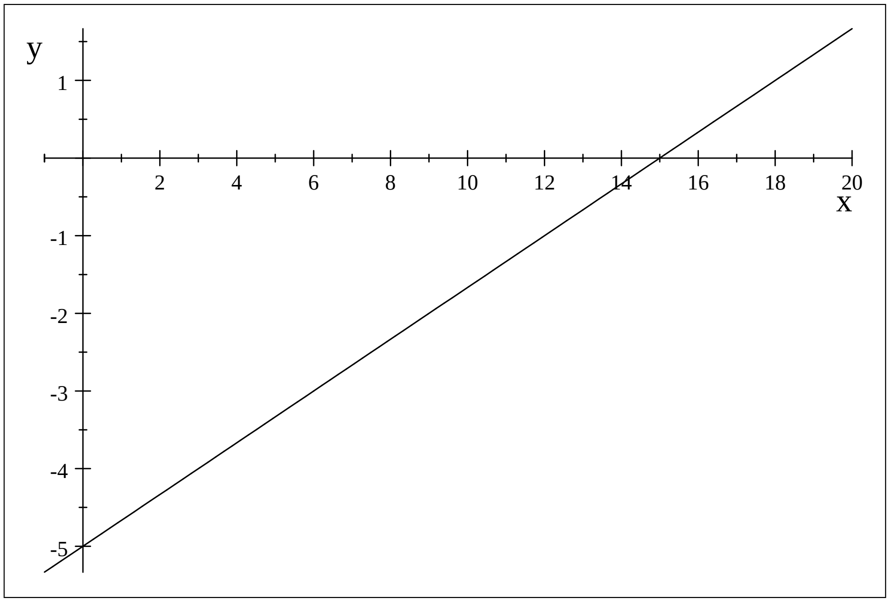
1. The x intercept is obtained by letting $f(x) = 0$ and solving for x

$$\begin{aligned} \frac{1}{3}x - 5 &= 0 \\ \frac{1}{3}x &= 5 \\ x &= 15. \end{aligned}$$

Thus the x intercept is the point $(15, 0)$.The y intercept is obtained by computing $f(0)$,

$$\begin{aligned} f(0) &= \frac{1}{3}0 - 5 \\ &= -5. \end{aligned}$$

Thus, the y intercept is the point $(0, -5)$.



2. Find the slopes of the lines passing through $(-3,-2)$ and $(-1,8)$. We first start by computing the slope

$$\begin{aligned} m &= \frac{8+2}{-1+3} \\ &= 5. \end{aligned}$$

By the point-slope formula is obtained by:

$$\begin{aligned} y-(-2) &= 5(x-(-3)) \\ y+2 &= 5(x+3) \\ y &= 5(x+3)-2 \\ &= 5x+13. \end{aligned}$$

3. Given the line L with equation $5x-2y=6$, find the equation of the line passing through the point $(0,1)$ and perpendicular to L. We first compute the slope of the line $5x-2y=6$,

$$\begin{aligned} -2y &= -5x+6 \\ y &= \frac{-5}{-2}x+\frac{6}{-2} \\ &= \frac{5}{2}x-\frac{1}{3}. \end{aligned}$$

The slope of the line L is $5/2$. The slope of any line perpendicular to it must be $-2/5$. Thus, by the point slope formula, we obtain

$$\begin{aligned} y-1 &= -\frac{2}{5}(x-0) \\ y &= -\frac{2}{5}x+1. \end{aligned}$$

Homework: Page $146/2, 9, 13, 19, 23, 29, 31, 37, 40, 41, 47, 48, 50, 63, 65, 75.$

Section 2.2
Linear Equations and Models
Lecture notes

Examples

1. Solve the following equation $3(x+2) - 3\,(x-1) = 2x+2$
2. Solve the following equation $2(x+1) + 3(2-x) = 8-x$
3. Solve the following equation $(x+2)(x-3) = (x-4)(x+5)$
4. Solve the following equation $\frac{2x}{x-3} = 7 + \frac{6}{x-3}$
5. Find three consecutive even integers so that the first plus twice the second is twice the third

Solutions

1. The solution set is $\left\{\frac{7}{2}\right\}$

$$\begin{aligned} 3(x+2) - 3\,(x-1) &= 2x+2 \\ 3x+6-3x+3 &= 2x+2 \\ 9 &= 2x+2 \\ 2x+2 &= 9 \\ 2x &= 7 \\ x &= 7/2 \end{aligned}$$

2. The solution is the set of all real numbers, namely, $\mathbb{R}$

$$\begin{aligned} 2(x+1) + 3(2-x) &= 8-x \\ 2x+2+6-3x &= 8-x \\ -x+8 &= 8-x \\ -x+x &= 8-8 \\ 0x &= 0. \end{aligned}$$

3. Solution is: 7

$$(x+2)(x-3) = (x-4)(x+5)$$

4.

$$\begin{aligned}
\frac{2x}{x-3} &= 7+\frac{6}{x-3}\\
\frac{2x}{x-3}-\frac{6}{x-3} &= 7\\
\frac{2x-6}{x-3} &= 7\\
2x-6 &= 7(x-3)\\
2x-6 &= 7x-21\\
2x-7x &= -21+6\\
-5x &= -15\\
x &= \frac{-15}{-5}\\
x &= 3.
\end{aligned}$$

However, the equation is not defined when $x = 3$. Thus, we say the solution set is empty and we write $\emptyset$.

5. Let a, b, c be 3 consecutive even integers

$$\begin{aligned}
b &= a+2\\
c &= b+2 = a+2+2 = a+4.
\end{aligned}$$

Also, since the first plus twice the second is twice the third

$$\begin{aligned}
a+2b &= 2c\\
a+2\,(a+2) &= 2\,(a+4)\\
a+2a+4 &= 2a+8\\
3a+4 &= 2a+8\\
3a-2a &= 8-4\\
a &= 4\\
b &= 6\\
c &= 8.
\end{aligned}$$

Homework Page $168/9, 13, 15, 19, 21, 29, 36, 41, 47, 58, 73, 77$.

2.3 Quadratic Functions

Definition 1 *A* ***quadratic function*** *(written in general form) is a function of the type*

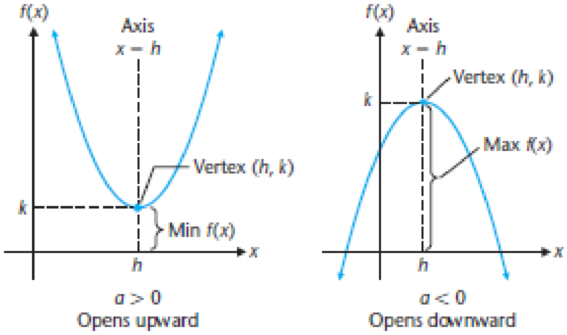
$$f(x) = ax^2 + bx + c$$

where a is a nonzero real value. A quadratic is also called a polynomial of degree 2. The graph of a quadratic is called a "parabola".

Definition 2 *Vertex form. A quadratic function could also be written as*

$$f(x) = a(x-h)^2 + k$$

such that the point (h, k) is the vertex of the parabola, and the axis of symmetry is $x = h$



Definition 3 *Square completion. To complete the square of the quadratic expression*

$$x^2 + bx$$

we need to add $\left(\frac{b}{2}\right)^2$ to obtain a perfect square. In other other words,

$$\begin{aligned} x^2 + bx + \left(\frac{b}{2}\right)^2 &= x^2 + 2\frac{b}{2}x + \left(\frac{b}{2}\right)^2 \\ &= \left(x + \frac{b}{2}\right)^2. \end{aligned}$$

Definition 4 *The x-coordinates of the vertex of the parabola $f(x) = ax^2+bx+c$ is given by*

$$x = -\frac{b}{2a}.$$

Example 5 *Using graph transformations on $g(x) = x^2$, sketch the function $f(x) = 2(x-3)^2 + 4$*

Example 6 *Complete the square*

$$x^2 - \frac{3}{4}x$$

Example 7 *Find the vertex form of the parabolas* $f(x) = x^2 - 8x + 4, g(x) = -3x^2 + 8x - 5$

$$\begin{aligned} f(x) &= x^2 - 8x + 16 + 4 - 16 \\ &= (x-4)^2 + 4 - 16 \\ &= (x-4)^2 - 12 \end{aligned}$$

$$\begin{aligned} g(x) &= -3x^2 + 8x - 5 \\ &= -3\left(x^2 - \frac{8}{3}x\right) - 5 \\ &= -3\left(x^2 - \frac{8}{3}x + \left(\frac{8}{2*3}\right)^2\right) - 5 + 3\left(\frac{8}{2*3}\right)^2 \\ &= -3\left(x^2 - \frac{8}{3}x + \frac{16}{9}\right) - 5 + \frac{16}{3} \\ &= -3\left(x - \frac{4}{3}\right)^2 + \frac{1}{3}. \end{aligned}$$

Example 8 *The height of a projectile traveling through space is given a function of time* t *by*

$$h(t) = 200 - 16t^2$$

When will the projectile hit the ground.

$$\begin{aligned} h(t) &= 0 \textit{ implies } 200 - 16t^2 = 0 \\ -16t^2 &= -200 \\ 16t^2 &= 200 \\ t^2 &= \frac{200}{16} = \frac{25}{2} \\ t &= \pm\sqrt{\frac{25}{2}} \end{aligned}$$

Since time may not be negative, then $t = \sqrt{\frac{25}{2}} = 3.5355$ *seconds.*

Section 2.5 Quadratic equations and models

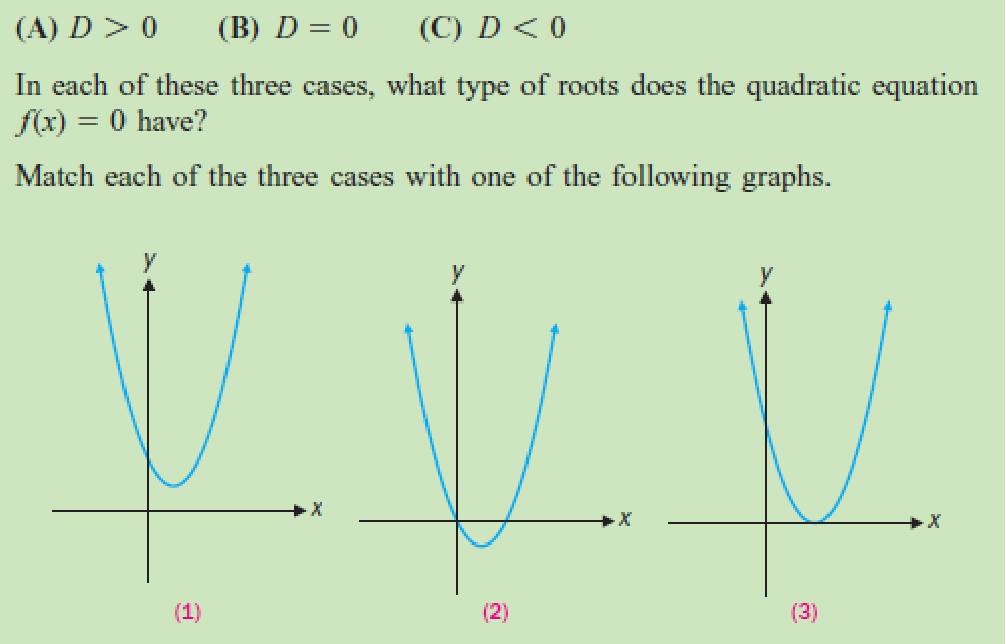
Theorem 1 *(Zero factor theorem) Given the equation* $m \cdot n = 0$*, then we must either have* $m = 0$ *or* $n = 0$.

Theorem 2 *(The quadratic formula) Given an equation of the type* $ax^2 + bx + c = 0$*, we obtain the solutions as follows*

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

such that $\Delta = b^2 - 4ac$ *is called the discriminant.*

The discriminant of an equation give us valuable information about the type and number of solutions for the equation



(A) $D > 0$ **(B)** $D = 0$ **(C)** $D < 0$

In each of these three cases, what type of roots does the quadratic equation $f(x) = 0$ have?

Match each of the three cases with one of the following graphs.

Theorem 3 *Given* $ax^2 + bx + c = 0$*, we have*

$$\begin{cases} \text{2 complex solutions if } \Delta < 0 \\ \text{2 real solutions if } \Delta > 0 \\ \text{A unique solution if } \Delta = 0 \end{cases}$$

Example 4 *Solve* $(x - 8)(2x + 3) = 0$

$$\begin{aligned} x - 8 &= 0 \text{ or } 2x + 3 = 0 \\ x &= 8 \text{ or } x = -\frac{3}{2} \end{aligned}$$

We say the solution set is $\{8, -\frac{3}{2}\}$.

Example 5 *Solve by factoring* $11x = 2x^2 + 12$

Solution is: $4, \frac{3}{2}$

Example 6 *Solve by Completeting the square* $y^2 - 10y - 3$

$$\begin{aligned} y^2 - 10y - 3 &= y^2 - 10y + 25 - 3 - 25 \\ &= \left(y^2 - 10y + 25\right) - 3 - 25 \\ &= (y-5)^2 - 28 \end{aligned}$$

$$\begin{aligned} (y-5)^2 - 28 &= 0 \\ (y-5)^2 &= 28 \\ y - 5 &= \pm\sqrt{28} \\ y &= \pm\sqrt{28} + 5 \\ &= 2\sqrt{7} + 5 \end{aligned}$$

Example 7 *Solve by using the quadratic formula* $7x^2 + 6x + 4 = 0$*, Solution is:* $\frac{1}{7}i\sqrt{19} - \frac{3}{7}, -\frac{1}{7}i\sqrt{19} - \frac{3}{7}$

$$\begin{aligned} a &= 7, b = 6, c = 4 \\ x &= \frac{-6 \pm \sqrt{36 - 4 \times 28}}{14} \\ &= -\frac{3}{7} + \frac{1}{7}i\sqrt{19} \end{aligned}$$

Example 8 *Use the discriminant to determine the number and type of zeros (solutions)* $0.3x^2 + 3.6x + 10.8 = 0$

$$\begin{aligned} a &= 0.3, b = 3.6, c = 10.8 \\ \Delta &= (3.6)^2 - (4\,(10.8)\,0.3) \\ &= 0 \end{aligned}$$

Thus, we have a unique solution.

Additional Equation-Solving Techniques

Example 1

1. Solve the equations

$$\begin{aligned}\sqrt{5x+6}+6 &= 0\\ \sqrt{5x+6} &= -6\end{aligned}$$

   We have no solution !

2. Solve algebraically $\sqrt{4x^2+8x+7}-x=1$, Solution is: $-1+i$

$$\begin{aligned}\sqrt{4x^2+8x+7}-x &= 1\\ \sqrt{4x^2+8x+7} &= x+1\\ \left(\sqrt{4x^2+8x+7}\right)^2 &= (x+1)^2\\ 4x^2+8x+7 &= x^2+2x+1\\ 4x^2-x^2+8x-2x+7-1 &= 0\\ 3x^2+6x+6 &= 0\\ x^2+2x+2 &= 0\\ x &= \frac{-2\pm\sqrt{4-8}}{2}\\ &= \frac{-2\pm 2i}{2}\\ &= -1\pm i\end{aligned}$$

3. Solve the equation $\sqrt{2x+3}=\sqrt{x^2-12}$

$$\begin{aligned}\sqrt{2x+3} &= \sqrt{x^2-12}\\ \left(\sqrt{2x+3}\right)^2 &= \left(\sqrt{x^2-12}\right)^2\\ 2x+3 &= x^2-12\\ x^2-2x-12-3 &= 0\\ x^2-2x-15 &= 0\\ x &= \frac{2\pm\sqrt{4+4(15)}}{2}\\ x &= \frac{2\pm 8}{2}\\ x &= 5,-3\end{aligned}$$

4. Solve the equation $y^{\frac{1}{2}}-3y^{\frac{1}{4}}+2=0$. By substitution, we let $x=y^{\frac{1}{2}}$, then

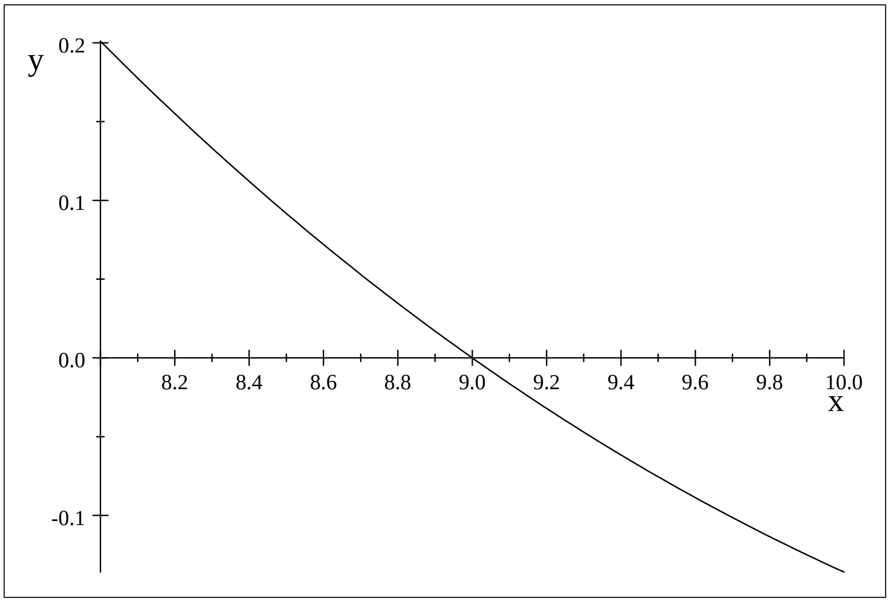
$x^{\frac{1}{2}} = \left(y^{\frac{1}{2}}\right)^{\frac{1}{2}} = y^{\frac{1}{4}}$

$$\begin{aligned}
y^{\frac{1}{2}} - 3y^{\frac{1}{4}} + 2 &= 0 \\
x - 3x^{\frac{1}{2}} + 2 &= 0 \\
x - 3\sqrt{x} + 2 &= 0 \\
-3\sqrt{x} &= -x - 2 \\
3\sqrt{x} &= x + 2 \\
\sqrt{x} &= \frac{x+2}{3} \\
x &= \left(\frac{x+2}{3}\right)^2 \\
&= \frac{x^2+4x+4}{9} \\
\frac{1}{9}x^2 + \frac{4}{9}x + \frac{4}{9} - x &= 0 \\
x^2 + 4x + 4 - 9x &= 0 \\
x^2 - 5x + 4 &= 0
\end{aligned}$$

$$\begin{aligned}
x &= \frac{5+\sqrt{25-16}}{2}, x = \frac{5-\sqrt{25-16}}{2} \\
x &= 4, x = 1
\end{aligned}$$

Thus, $x = y^{\frac{1}{2}}$ implies that $y = 16, 1$.

5. Solve by graphing the equation $m - 7\sqrt{m} + 12$



According to the graph, the solution is $m = 9$.

2.6
Solving Inequalities

Theorem 1

For $p > 0$

1. $|ax + b| < p$ is equivalent to $-p < ax + b < p$.

2. $|ax + b| = p$ is equivalent to $ax + b = p$ or $ax + b = -p$.

3. $|ax + b| > p$ is equivalent to $ax + b < -p$ or $ax + b > p$.

Theorem 2 *The Location Theorem*

If $f(x)$ is a continuous function and $f(a)$ and $f(b)$ have opposite signs, then there must be a zero of f somewhere between $x = a$ and $x = b$.*

Example 3

1. Solve the following inequality $|s - 5| < 3$

$$\begin{aligned} -3 &< s - 5 < 3 \\ -3 + 5 &< s < 3 + 5 \\ 2 &< s < 8 \end{aligned}$$

The solution set is $(2, 8)$

2. Solve the inequality $|t - 3| > 4$

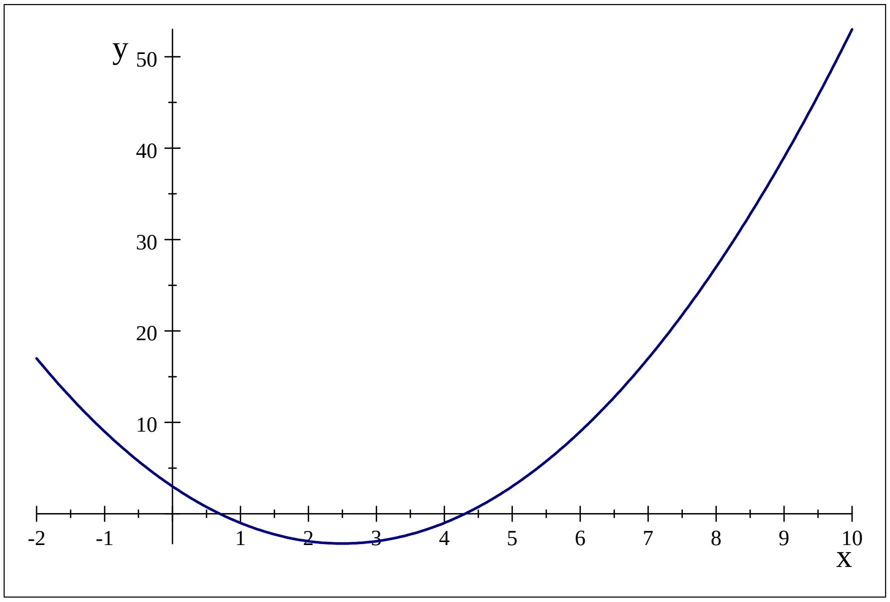
$$\begin{aligned} t - 3 &> 4 \text{ or } t - 3 < -4 \\ t &> 7 \text{ or } t < -1 \end{aligned}$$

The solution set is the set $(7, \infty) \cup (-\infty, -1)$

3. Solve the inequality $x^2 - 5x + 3 > 0$

$$\begin{aligned} \text{First we solve the equation } x^2 - 5x + 3 &= 0 \\ x &= \frac{5 - \sqrt{25 - 12}}{2}, x = \frac{5 + \sqrt{25 - 12}}{2} \\ x &= \frac{5}{2} - \frac{1}{2}\sqrt{13}, x = \frac{5}{2} + \frac{1}{2}\sqrt{13} \end{aligned}$$

Thus, $x^2 - 5x + 3 = \left(x - \left(\frac{5}{2} - \frac{1}{2}\sqrt{13}\right)\right)\left(x - \left(\frac{5}{2} + \frac{1}{2}\sqrt{13}\right)\right)$. Let us look graph of the quadratic



Identify when the zeros are located, the solution to the inequality $x^2 - 5x + 3 > 0$ is $\left(-\infty, \frac{5}{2} - \frac{1}{2}\sqrt{13}\right) \cup \left(\frac{5}{2} + \frac{1}{2}\sqrt{13}, \infty\right)$.

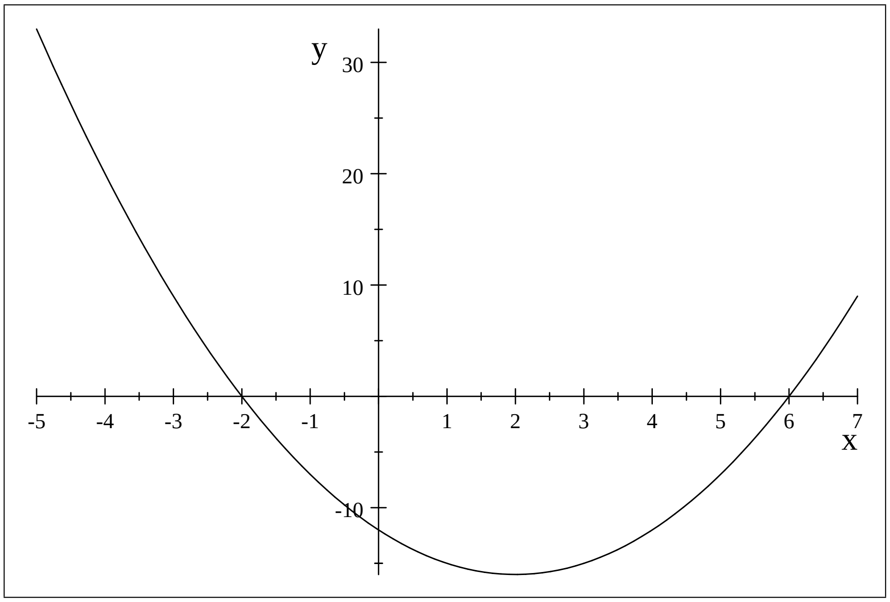
4. Solve the inequality, $x^2 < 4x + 12$

$$\begin{aligned} x^2 &< 4x + 12 \\ x^2 - 4x - 12 &< 0 \end{aligned}$$

We first solve the equation

$$\begin{aligned} x^2 - 4x - 12 &= 0 \\ x &= \frac{4 + \sqrt{16 + 48}}{2}, x = \frac{4 - \sqrt{16 + 48}}{2} \\ x &= 6, -2. \end{aligned}$$

Next, we plot $x^2 - 4x - 12$



The solution to the inequality $x^2 - 4x - 12 < 0$ is then $(-2, 6)$

5. The pages of a texbook have uniform margin of 2 centimeters on all four sides. If the area of the entire page is 480 and the areal of the printed portion is 320, find the dimension of the page. Let x, y be the dimensions of the page

$$\begin{aligned} xy &= 480, \\ (x-4)(y-4) &= 320 \end{aligned}$$

Since $y = \frac{480}{x}$, then

$$\begin{aligned} (x-4)(y-4) &\Rightarrow (x-4)(\frac{480}{x} - 4) = 320 \\ &\Rightarrow 496 - \frac{1920}{x} - 4x = 320 \\ &\Rightarrow -\frac{1920}{x} - 4x = 320 - 496 \\ &\Rightarrow -\frac{1920}{x} - 4x = -176 \\ &\Rightarrow \frac{1920}{x} + 4x = 176 \\ &\Rightarrow 1920 + 4x^2 = 176x \\ &\Rightarrow 4x^2 - 176x + 1920 = 0 \\ &\Rightarrow x = 24, 20 \end{aligned}$$

Section 3.1 (Polynomial Functions and Models)

A polynomial is the finite sum of terms of the form ax^k such that $a \in \mathbb{R}$, and k is a whole number. So in general, a polynomial

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

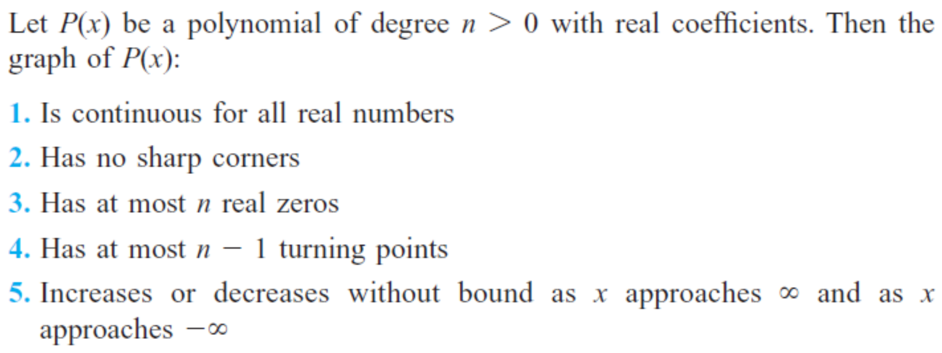
One can see that

a. the function $f(x) = 2x - 5x + (2 + i)\, x^{12}$ is a polynomial

b. the function $g(x) = 3\sqrt{x} + 11x^{45}$ is not a polynomial

Definition 1 *(Zeros or roots) If* $p(x)$ *is a function, and* $p(r) = 0$ *then* r *is called the root of the function* $p(x)$.

Theorem 2 *Properties of graph of polynomials*



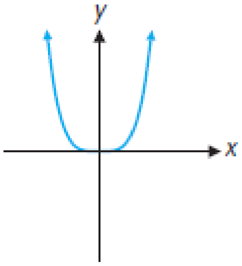
Let $P(x)$ be a polynomial of degree $n > 0$ with real coefficients. Then the graph of $P(x)$:

1. Is continuous for all real numbers
2. Has no sharp corners
3. Has at most n real zeros
4. Has at most $n - 1$ turning points
5. Increases or decreases without bound as x approaches ∞ and as x approaches $-\infty$

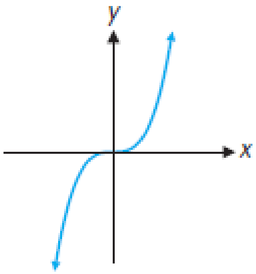
Theorem 3 *Left and right end behaviors of graphs of polynomials*

Let $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial function with real coefficients, $a_n \neq 0$, $n > 0$. If P has:

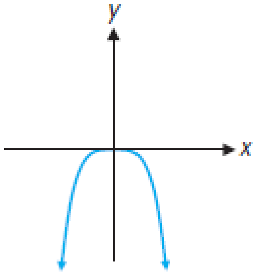
1. Positive leading coefficient, even degree: $P(x) \to \infty$ as $x \to \infty$ and $P(x) \to \infty$ as $x \to -\infty$ (like the graph of $y = x^2$).
2. Positive leading coefficient, odd degree: $P(x) \to \infty$ as $x \to \infty$ and $P(x) \to -\infty$ as $x \to -\infty$ (like the graph of $y = x^3$).
3. Negative leading coefficient, even degree: $P(x) \to -\infty$ as $x \to \infty$ and $P(x) \to -\infty$ as $x \to -\infty$ (like the graph of $y = -x^2$).
4. Negative leading coefficient, odd degree: $P(x) \to -\infty$ as $x \to \infty$ and $P(x) \to \infty$ as $x \to -\infty$ (like the graph of $y = -x^3$).



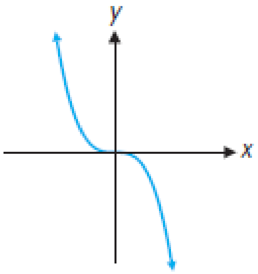
Case 1



Case 2



Case 3

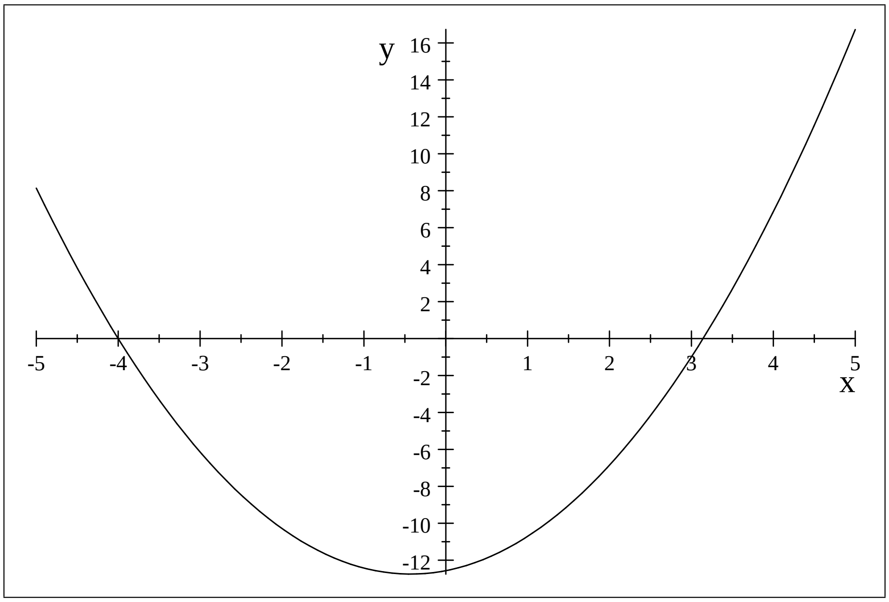


Case 4

Example 4 *Find all the zeros of the polynomials* $P(x) = x\left(x^2-9\right)\left(x^2+4\right)$,
Solution is: $-3, 3, -2i, 2i, 0$

Example 5 *Find all the zeros of the polynomials* $P(x) = \left(x^2-5x+6\right)\left(x^2-5x+7\right)$,
Solution is: $\frac{1}{2}i\sqrt{3}+\frac{5}{2}, \frac{5}{2}-\frac{1}{2}i\sqrt{3}, 3, 2$

Example 6 *Sketch the graph of a polynomial which has for zeros* $x = -4, \pi$; *even degree, positive leading coefficient.*



Example 7 *A rectangular storage container measuring 2 feet 2 feet by 3 feet is coated with protective coating of plastic of uniform thickness. Find the volume of plastic as a function of the thickness x of the coating*

If we consider the side of dimension 2×2, its volume is $4x$. If we consider the side which is 2×3, we obtain $6x$. Since we have 2 versions of the first one and 4 versions of the second, the volume obtained is

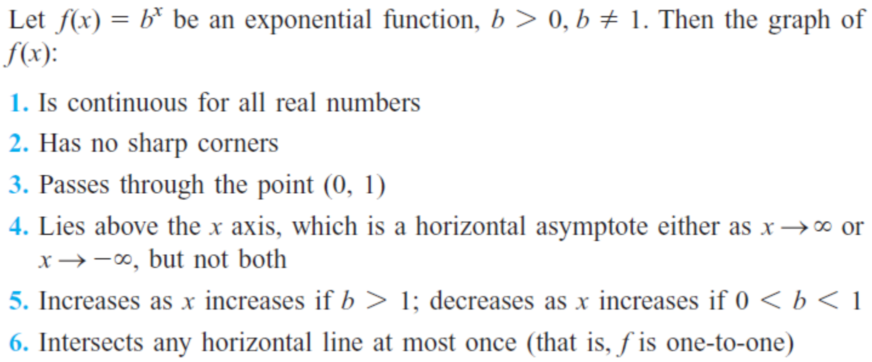
$$V = 2(4x) + 4(6x) = 32x$$

Section 4.1 Exponential Function

Definition 1

The equation $f(x) = b^x, \dot{b} > 0, b \neq 1$ defines an exponential function. b is called the base of the exponential.

Theorem 2 *Properties of Graphs of exponential functions*



Let $f(x) = b^x$ be an exponential function, $b > 0, b \neq 1$. Then the graph of $f(x)$:

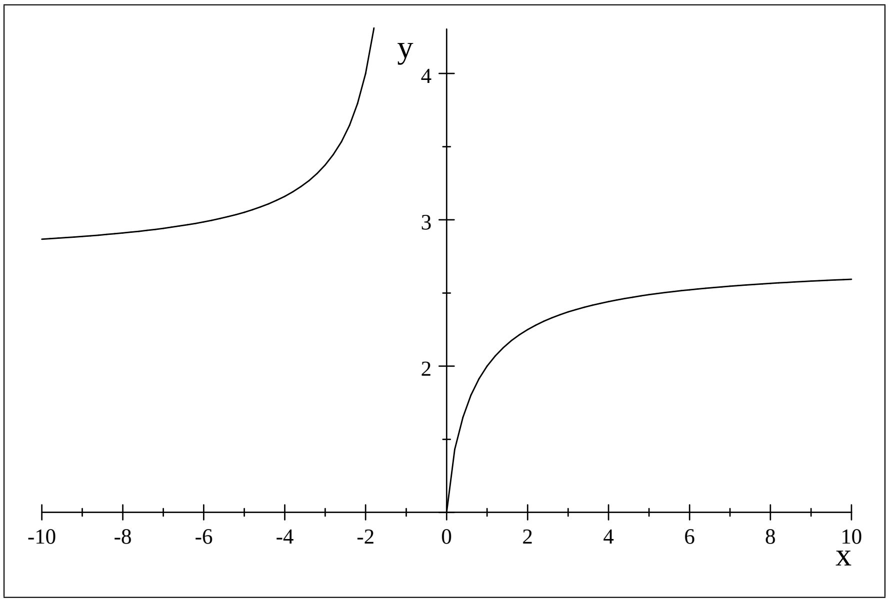
1. Is continuous for all real numbers
2. Has no sharp corners
3. Passes through the point (0, 1)
4. Lies above the x axis, which is a horizontal asymptote either as $x \to \infty$ or $x \to -\infty$, but not both
5. Increases as x increases if $b > 1$; decreases as x increases if $0 < b < 1$
6. Intersects any horizontal line at most once (that is, f is one-to-one)

Theorem 3 *Exponential properties*

1. $a^x a^y = a^{x+y}$
2. $(a^x)^y = a^{xy}$
3. $(ab)^x = a^x b^x$
4. $\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$
5. $\frac{a^x}{a^y} = a^{x-y}$
6. $a^x = a^y \Leftrightarrow x = y$
7. $a^x = b^x \Leftrightarrow a = b$

Definition 4 *By definition, the irrational number e called exponential is*

$$\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x = e$$



graph of $\left(1+\frac{1}{x}\right)^x$

Definition 5

The function $f(x) = e^x$ is called the exponential function with base e

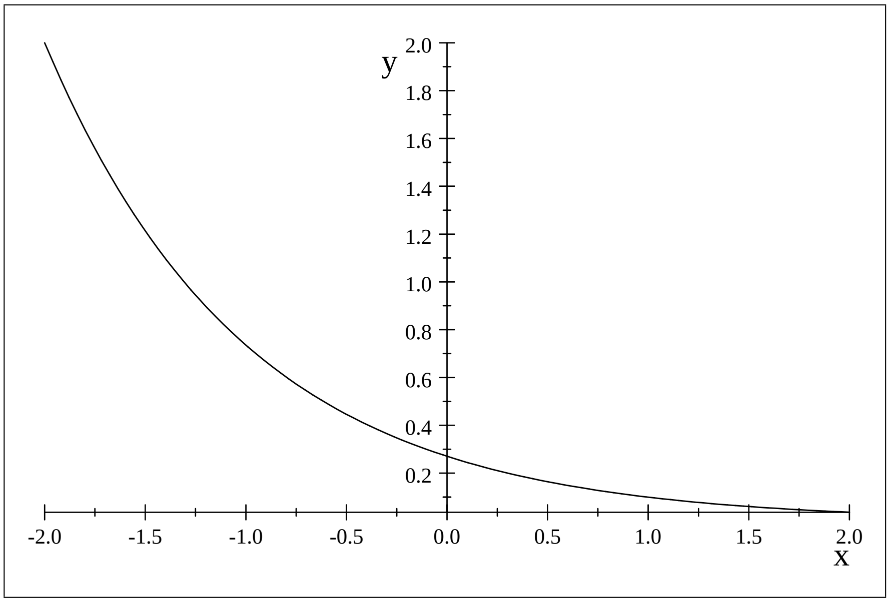
Example 6

1. Simplify the followings $\frac{e^{4-3x}}{e^{2-5x}}, 10^{3x-1}10^{4-x}$
2. Graph the function $g(x) = 2e^{-(x+2)}$ by using the appropriate transformations
3. Solve the equation $7^{x^2} = 7^{2x+3}$
4. Solve the equation $x^2e^x - 5xe^x = 0$
5. Find all numbers a such that $a^2 = a^{-2}$

Solution 7

1. Will be shown in class

2.



3. The exponents must be the same, thus,

$$\begin{aligned} x^2 &= 2x+3 \\ x^2-2x-3 &= 0 \end{aligned}$$

   Solution is: $3, -1$

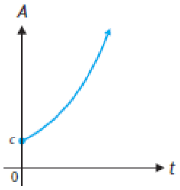
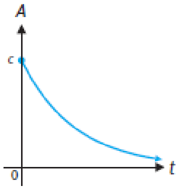
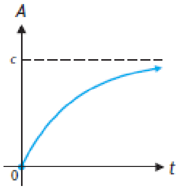
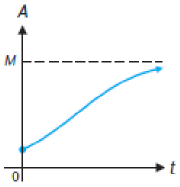
4.

$$\begin{aligned} x^2e^x - 5xe^x &= 0 \\ e^x\left(x^2-5x\right) &= 0 \\ e^x &= 0 \text{ or } x^2-5x \end{aligned}$$

   However, the exponential function has no zero since it never touch the x-axis. The only solutions obtained are from the second equation $x^2-5x=0$

$$\begin{aligned} x^2-5x &= 0 \\ x\left(x-5\right) &= 0 \\ x &= 0, 5 \end{aligned}$$

Section 4.2 Exponential Models

Unlimited growth	$A = A_0e^{kt}$ $k > 0$		Short-term population growth (people, bacteria, etc.); growth of money at continuous compound interest
Exponential decay	$A = A_0e^{-kt}$ $k > 0$		Radioactive decay; light absorption in water, glass, and the like; atmospheric pressure; electric circuits
Limited growth	$A = c(1 - e^{-kt})$ $c, k > 0$		Learning skills; sales fads; company growth; electric circuits
Logistic growth	$A = \dfrac{M}{1 + ce^{-kt}}$ $c, k, M > 0$		Long-term population growth; epidemics; sales of new products; spread of rumors; company growth

Population growth

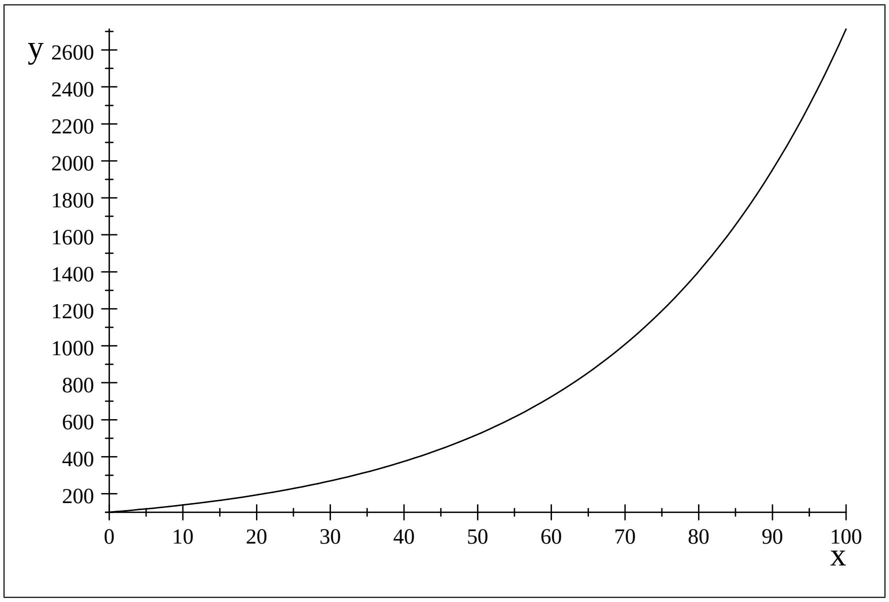
Mexico has a population of around 100 million people, and it is estimated that the population will double in 21 years. If the population continues to grow at the same rate. What will the population be in 15 years?

$$\begin{aligned} A &= A_0e^{kt} \\ A_0 &= 100 \\ 200 &= 100e^{21k} \\ e^{21k} &= 2 \Rightarrow 21k = \ln 2 \\ &\Rightarrow \quad k = \frac{\ln 2}{21} = 3.3007 \times 10^{-2} \end{aligned}$$

In 15 years,

$$\begin{aligned} A &= 100e^{15k} \\ &= 100e^{15\left(\frac{\ln 2}{21}\right)} \\ &\approx 164.07 \end{aligned}$$

Here is the graph of the growth of the population $A(t) = 100e^{\left(\frac{\ln 2}{21}\right)t}$



Radioactive decay

The radioactive isotope gallium 67 used in the diagnosis of malignant tumors has a biological half-life of 46.5 hours. If we start with 100 milligrams of the isotope, how many milligrams will be left after 1 week

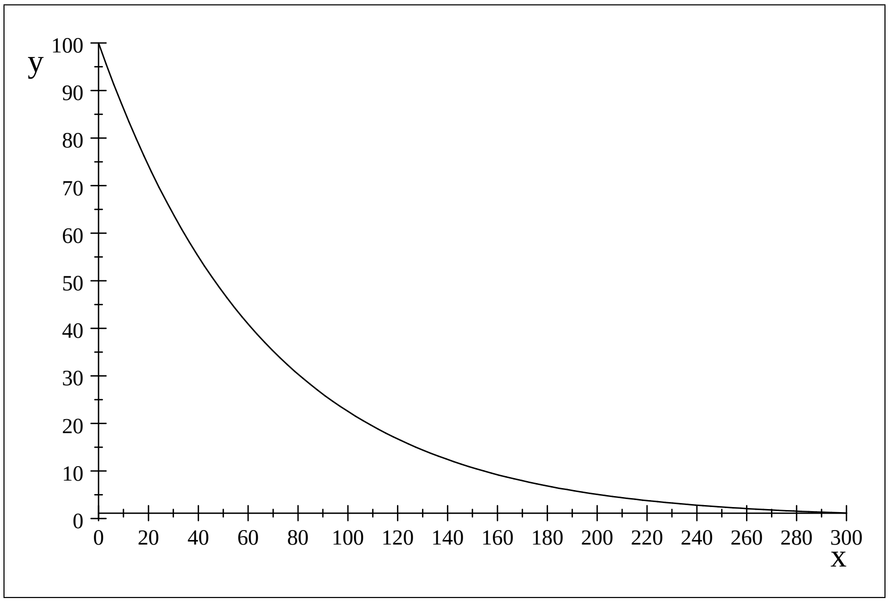
$$\begin{aligned} A(t) &= A_0e^{-kt} \\ A(t) &= 100e^{-kt} \end{aligned}$$

$$\begin{aligned} 50 &= 100e^{-46.5k} \\ A(t) &= A_0e^{-kt}e^{-46.5k} = 0.5 \\ -46.5k &= \ln(0.5) \\ k &= -\frac{\ln(0.5)}{46.5} \end{aligned}$$

In a week, we have $(24)\,7 = 168$ hours, thus,

$$\begin{aligned} A\,(168) &= 100e^{\frac{\ln(0.5)}{46.5}168} \\ &= 8.1735 \end{aligned}$$

Here is the graph of $A(t) = 100e^{\left(\frac{\ln(0.5)}{46.5}\right)t}$



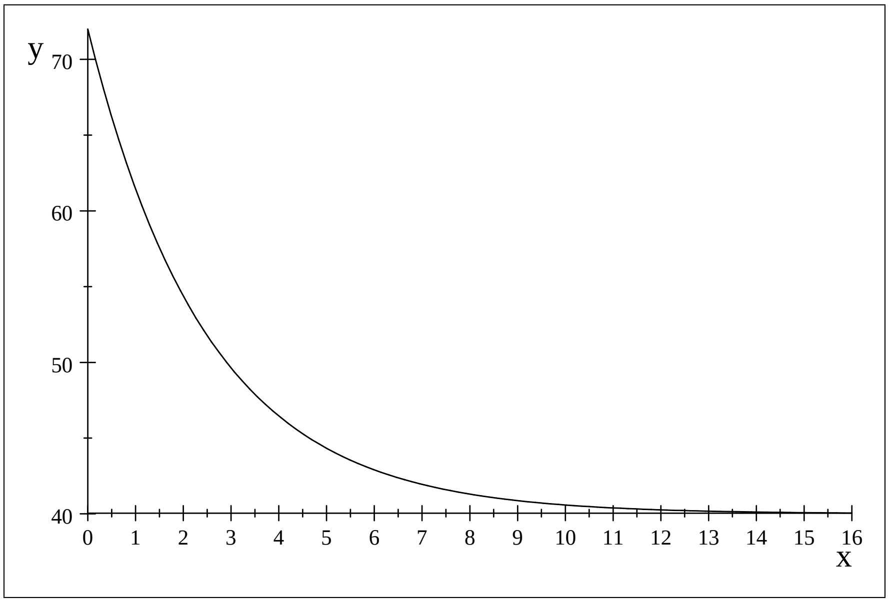
Newton's Law of cooling

This law states that the rate at which an object cools is proportional to the difference in temperature between the object and its surrounding medium. The temperature T of an object t hours later is given by

$$T = T_m + (T_0 - T_m)\, e^{-kt}$$

T_m is the temperature of the surrounding, T_0 is the initial temperature of the object. Suppose a bottle of wine at a room temperature of 72 degree is placed in the refrigerator to cool off. If the temperature in the refrigerator is kept at 40 degree and $k = 0.4$, find the temperature of the wine after 3 hours

$$\begin{aligned} T(t) &= 40 + (72 - 40)\, e^{-0.4t} \\ &= 32e^{-0.4t} + 40 \end{aligned}$$



Temperature T

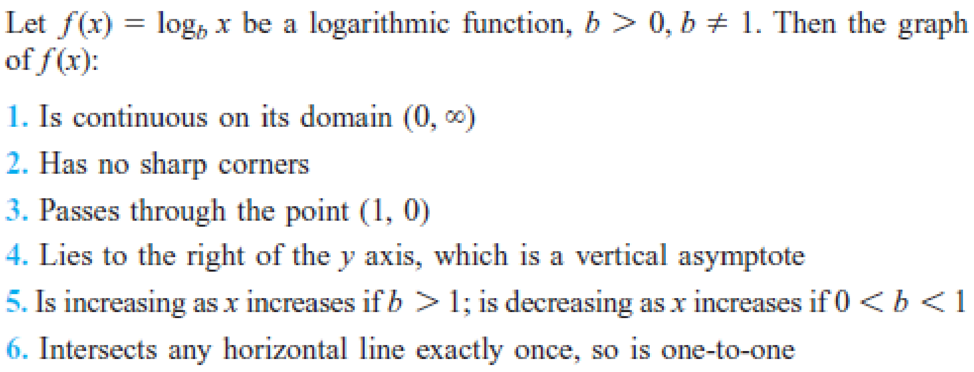
$$\begin{aligned} T(3) &= 32e^{-0.4(3)} + 40 \\ &= 49.638 \end{aligned}$$

4.3 Log functions

a By definition $y = \log_b x$ if and only if $b^y = x$

b The inverse function of $\log_b x$ is b^x

c



Let $f(x) = \log_b x$ be a logarithmic function, $b > 0$, $b \neq 1$. Then the graph of $f(x)$:

1. Is continuous on its domain $(0, \infty)$
2. Has no sharp corners
3. Passes through the point $(1, 0)$
4. Lies to the right of the y axis, which is a vertical asymptote
5. Is increasing as x increases if $b > 1$; is decreasing as x increases if $0 < b < 1$
6. Intersects any horizontal line exactly once, so is one-to-one

d

$$\begin{aligned}
\log_b 1 &= 0 \\
\log_b b &= 1 \\
\log_b b^x &= x \\
b^{\log_b x} &= x, x > 0 \\
\log_b MN &= \log_b M + \log_b N \\
\log_b \frac{M}{N} &= \log_b M - \log_b N \\
\log_b M^n &= n \log_b M
\end{aligned}$$

e $\log_e(x)$ is called the natural log of x and denoted $\ln x$.

Examples

1. Write in Exponential form

$$\begin{aligned}
\log_{10} 0.001 &= -3 \\
\log_2 \frac{1}{64} &= -6
\end{aligned}$$

2. Simplify the following expressions

$$\log_{25} 1$$
$$\log_{10} 10^5$$
$$\log_{1/5}\left(\frac{1}{25}\right)$$

3. Rewrite the expression in terms of $\log x$ and $\log y$

$$\log\left(\frac{x}{y}\right)$$
$$\log\left(x^4y^3\right)$$
$$\log\left(\frac{x^2}{\sqrt{y}}\right)$$

4. Rewrite the following as a single log

$$2\ln x + 5\ln y - \ln z$$

5. Find f^{-1}

$$f(x) = 2\log_2(x-5)$$

Solutions

1.

$$\begin{aligned}\log_{10} 0.001 &= -3\\ 10^{-3} &= 0.001\\ \log_2 \frac{1}{64} &= -6\\ 2^{-6} &= 1/64\end{aligned}$$

2.

$$\log_{25} 1 = 0$$
$$\log_{10}\left(10^5\right) = 5\log_{10} 10 = 5 \times 1 = 5$$
$$\log_{1/5}\left(\frac{1}{25}\right) = \log_{1/5}\left(\left(\frac{1}{5}\right)^2\right) = 2\log_{1/5}(1/5) = 2$$

3.

$$\log\left(\frac{x}{y}\right) = \log x - \log y$$

$$\begin{aligned}&\log\left(x^4y^3\right) = \log\left(x^4\right) + \log\left(y^3\right)\\ =\ &4\log x + 3\log y\end{aligned}$$

$$\begin{aligned}\log\left(\frac{x^2}{\sqrt{y}}\right) &= \log\left(x^2\right) - \log\left(\sqrt{y}\right)\\ &= 2\log(x) - (1/2)\log y\end{aligned}$$

4.

$$\begin{aligned} 2\ln x + 5\ln y - \ln z &= \ln\left(x^2\right) + \ln\left(y^5\right) - \ln z \\ &= \ln\left(x^2 y^5 z\right). \end{aligned}$$

5. Put

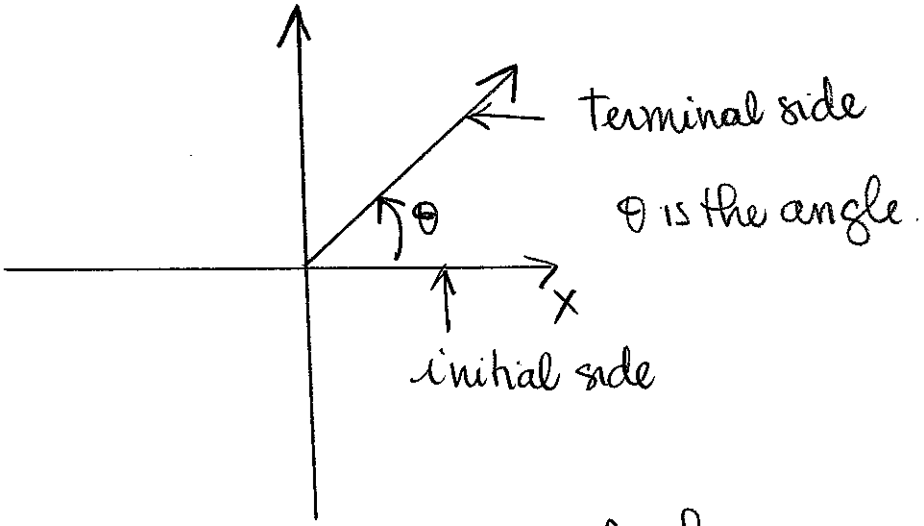
$$\begin{aligned} y &= 2\log_2(x-5) \\ y &= \log_2\left((x-5)^2\right) \\ 2^y &= (x-5)^2 \\ x-5 &= \sqrt{2^y} \\ x &= \sqrt{2^y}+5 \end{aligned}$$

Thus $f^{-1}(x) = \sqrt{2^x} + 5$

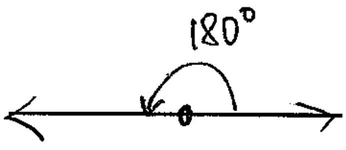
5.1 Angles and their Measures

An angle is formed by rotating a ray m called the initial side of the angle. The ray obtained after rotation is called the terminal side. The common point of the rays is called the vertex.

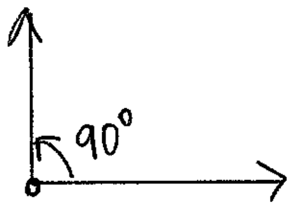
A positive angle is obtained by rotating a ray counterclockwise while a negative angle is obtained by rotating clockwise.



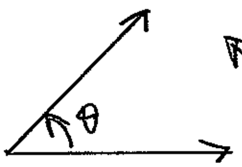
Definition A complete rotation yields an angle of measure 360°.



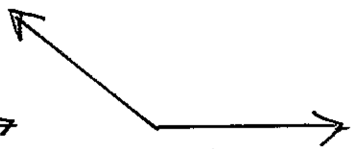
straight angle



right angle



acute angle



obtuse angle.

2 angles are complementary if their sum is 90°

2 angles are supplementary if their sum is 180°

1 degree = 60 min

1 min = 60 sec

<u>Example 1</u> Convert $21°27'12''$ to decimal degrees

$$1 \text{ degree} \rightarrow 60 \text{ min}$$

$$x \text{ degree} \rightarrow 27 \text{ min}$$

$$x \text{ degree} = \frac{27 \text{ min} \times 1 \text{ degree}}{60 \text{ min}} = \frac{27}{60} = 0.45°$$

$$1 \text{ degree} \rightarrow 3600 \text{ sec}$$

$$y \text{ degree} \rightarrow 12 \text{ sec}$$

$$y \text{ degree} = \frac{12 \cancel{\text{sec}} \times 1 \text{ deg}}{3600 \times \cancel{\text{sec}}} = \frac{12}{3600} \text{ degree} = 0.0033°$$

Thus $21°27'12'' = 21° + 0.45° + 0.0033°$

$$= 21.4533°$$

Example 2) Convert 105.183° in degree-minute-second form

$$105.183^\circ = 105 + 0.183^\circ$$

$$1^\circ \rightarrow 60'$$

$$0.183^\circ \rightarrow x$$

$$x = \frac{0.183^\circ \times 60'}{1^\circ} = 10.98'$$

Thus $105.183^\circ = 105^\circ (10.98)'$

$$10.98' = 10' + 0.98'$$

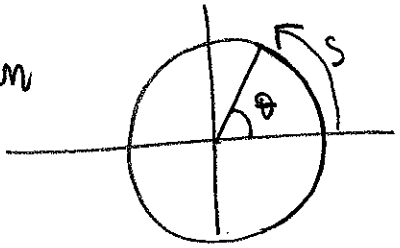
$$0.98' \approx 59''.$$

Thus $105.183^\circ = 105^\circ\, 10'\, 59''$.

Radian Measure

A positive angle θ has measure 1 radian if the length s of the arc opposite θ is equal to the radius.

$\theta = \frac{s}{r}$ radian

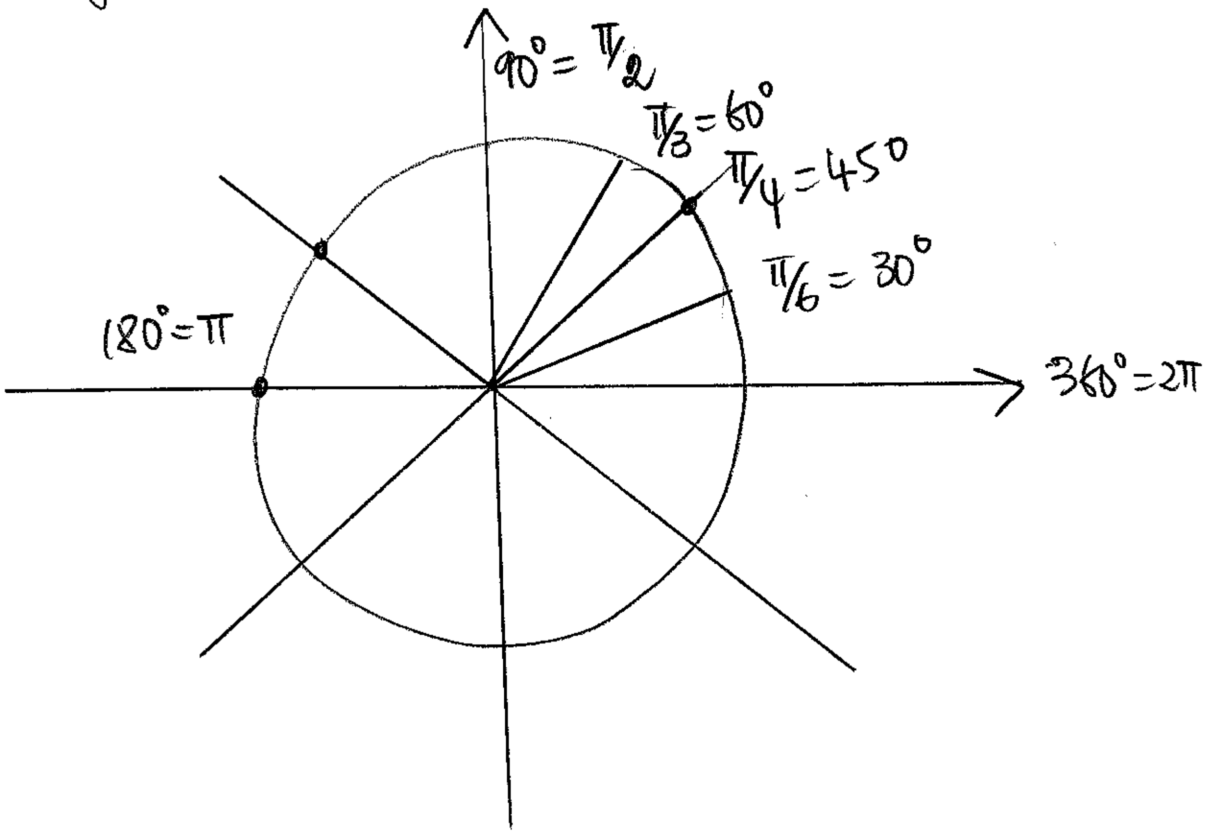


Example

Assume $\theta = 360°$, then $s = 2\pi r$.

thus in radian $\theta = \frac{2\pi r}{r} = 2\pi$ radian.

the flat angle has for radius $180° = \pi$ radian.



Example 1 Convert from degree to radian

$60°$ $\quad 180° \to \pi$ $\quad x = \frac{60\pi}{180} = \frac{\pi}{3}$ radian

$\quad\quad\quad 60° \to x$

$120°$ $\quad x = \frac{120\pi}{180} = \frac{12\pi}{18} = \frac{4\pi}{6} = \frac{2\pi}{3}$

$-150°$ $\quad x = \frac{-150\pi}{180} = \frac{-15\pi}{18} = \frac{-5\pi}{6}$

Example 2 Convert from radian to degree

$\frac{2\pi}{5}$ $\quad \pi \to 180°$ $\quad x = \frac{2\cancel{\pi} \times 180}{5\cancel{\pi}} = 72°$

$\quad\quad\quad \frac{2\pi}{5} \to x$

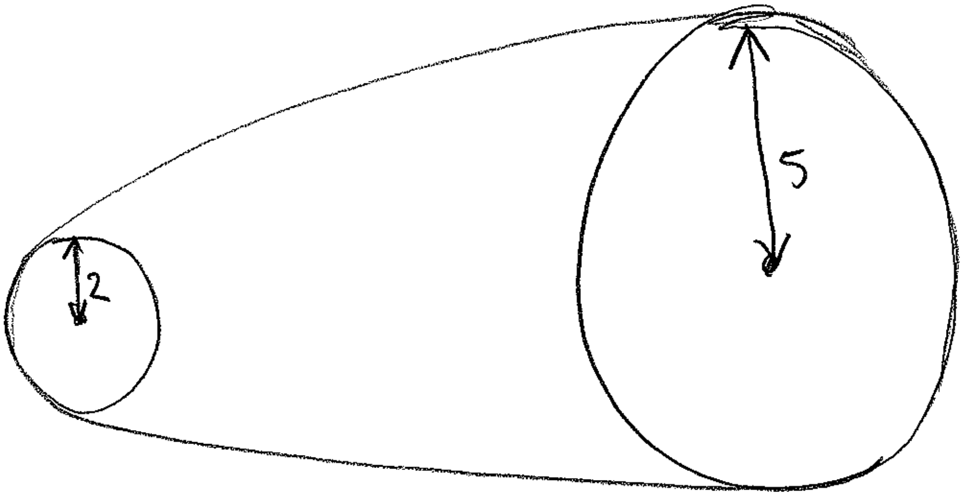
$-\frac{\pi}{2}$ $\quad x = -90°$

$\frac{6\pi}{5}$ $\quad \pi \to 180°$ $\quad x = \frac{6\pi \times 180}{5\pi} = 216°$

$\quad\quad\quad \frac{6\pi}{5} \to x$

Application

A belt connects a pulley of 2-inch radius with a pulley of 5-inch radius. If the smaller pulley turns around 20 radians, through how many radians will the Larger pulley turn.

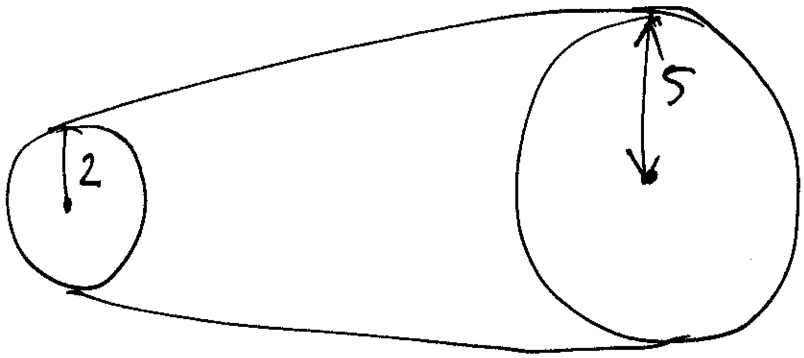


the length of cord pulled by the smaller pulley in 20 radians is

$$\theta = \frac{s}{r} \Rightarrow s = \theta \cdot r = 20r = 40$$

the same Length of cord is pulled by the Larger pulley

however $\theta' = \frac{s}{r'} \Rightarrow \theta' = \frac{40}{5} = 8$



If the smaller pulley turns around 20 times. Throug how many radians will the Larger pulley turn.

small: $2\pi r_1 = 2\pi(2) = 4\pi$

20 times $\rightarrow$ $20 \times 4\pi = 80\pi$.

big pulley: $2\pi r_2 = 10\pi$

$1 \text{ turn} \rightarrow 10\pi$

$x \rightarrow 80\pi$

$$x = \frac{80\pi \times 1 \text{ turn}}{10\pi} = 8 \text{ turns}$$

$$S = r_2\theta \rightarrow \theta = \frac{s}{r_2} = \frac{80\pi}{5} = 16\pi$$

Definition

Let us consider a particle moving at a constant speed along a circular arc of radius r.

s = length of the arc traveled in time t

- the Linear speed $= \dfrac{\text{arc length}}{\text{time}} = \dfrac{s}{t}$
- If θ is the angle of rotation, then the angular speed $= \dfrac{\theta}{t}$

<u>Example</u> A 15-inch diameter tire on a car makes 9.3 revolutions per second.

a) Find the angular speed of the tire in radians per sec

b) Find the speed of the car.

<u>Solution</u>

a) 9.3 revolution $\Longrightarrow$ $9.3(2\pi) = 18.6\pi$ rad/s = angular speed.

b) the Linear speed $= \dfrac{s}{t} = \dfrac{2\pi r \times (9.3)}{1} = \dfrac{2\pi \frac{15}{2} \times 9.3}{1}$

$= 438$ inches per sec

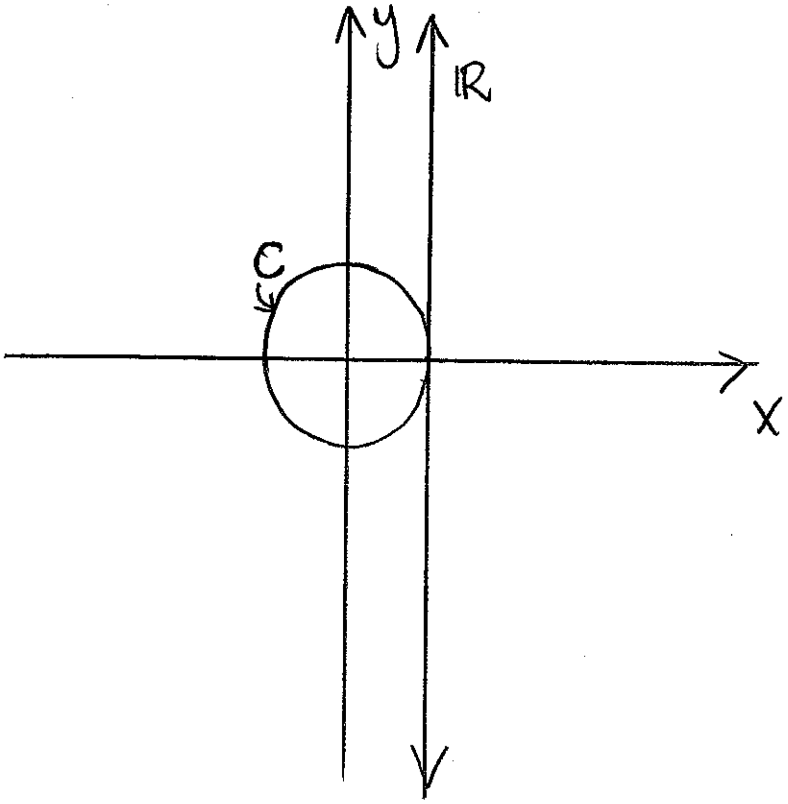
5.2 Trigonometric Functions A unit circle approach

Define the Wrapping function, $W: (-\infty, \infty) \to C$ where C is the unit circle. Such that

$W(\theta)$ = the point on the unit circle which has for radial angle θ.

For example $W(0) = (1,0)$, $W(\pi/2) = (0,1)$

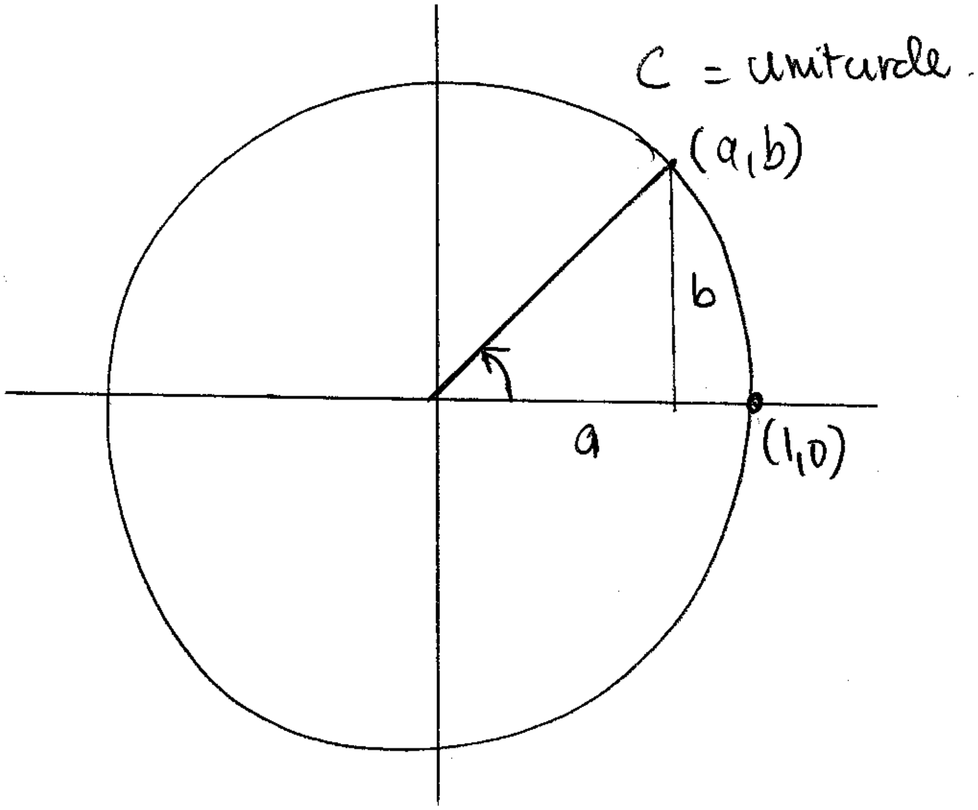
$W(\pi/4) = (\sqrt{2}/2, \sqrt{2}/2)$, $W(\pi) = (-1,0)$, $W(\frac{3\pi}{2}) = (0,-1)$.



Example Compute $W(3\pi)$, $W(-7\pi/2)$

$W\left(-\frac{7\pi}{2}\right) = (0,-1)$, $W(3\pi) = (-1,0)$

Trig functions



$\sin\theta = \frac{b}{1}$, $\cos\theta = \frac{a}{1}$

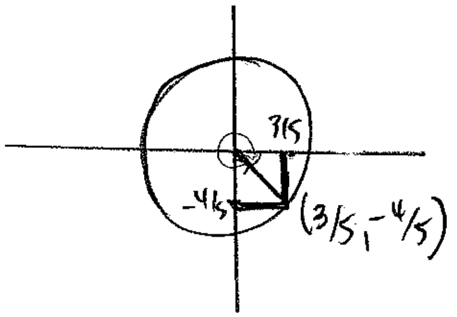
$\tan\theta = \frac{b}{a}$, $a \neq 0$

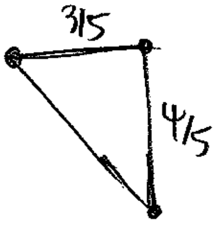
$\csc\theta = \frac{1}{b}$, $b \neq 0$

$\sec\theta = \frac{1}{a}$, $b \neq 0$

Memorize this

Example Find the values of all six trigonometric functions of the angle x if $W(x) = \left(\frac{3}{5}, -\frac{4}{5}\right)$





we check that $\sqrt{\left(\frac{3}{5}\right)^2+\left(\frac{-4}{5}\right)^2} = \sqrt{\frac{9+16}{25}} = \sqrt{\frac{25}{25}} = 1$

$\sin(x) = -\frac{4}{5}$, $\cos(x) = \frac{3}{5}$

$\tan(x) = \frac{-4/5}{3/5} = -\frac{4}{5} \times \frac{5}{3} = -\frac{4}{3}$

$\cot(x) = -\frac{3}{4}$, $\sec(x) = \frac{5}{3}$, $\csc(x) = -\frac{5}{4}$

Graphing sin(x)

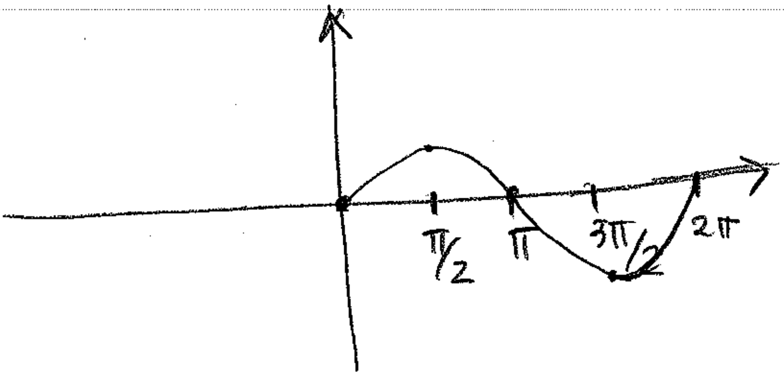
$\sin(0) = 0$

$\sin(\frac{\pi}{2}) = 1$

$\sin(\pi) = 0$

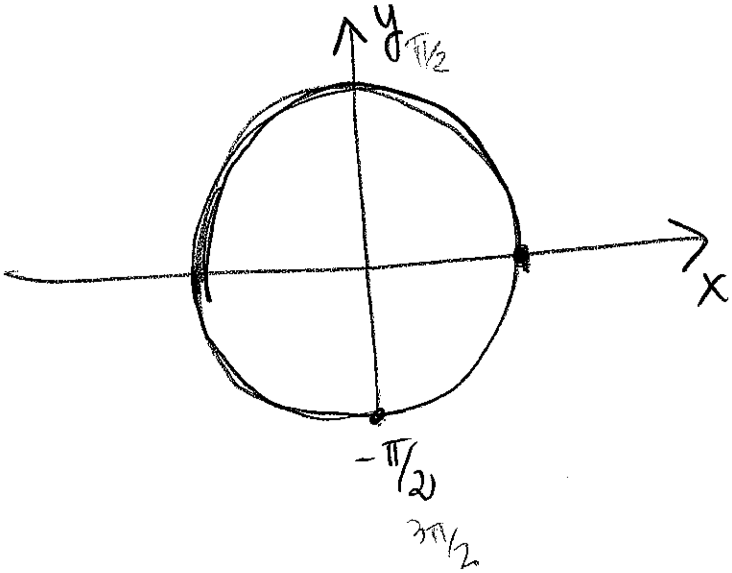
$\sin(\frac{3\pi}{2}) = -1$

$\sin(2\pi) = 0$



Example 1

Find all zeros and turning point of $y = \cos x$ on $[-\frac{\pi}{2}, \frac{3\pi}{2}]$



$x = -\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2} + \pi = \frac{3\pi}{2}$.

Worsheet 5.2

1. Compute $W(3\pi/2)$, $W(\frac{11\pi}{6})$, $W(-\frac{3\pi}{4})$

2. $\tan(\pi/6)$, $\cot(13\pi/4)$, $\sin(-10\pi/3)$

3. Find x if $W(x) = (-\frac{1}{2}, -\frac{\sqrt{3}}{2})$

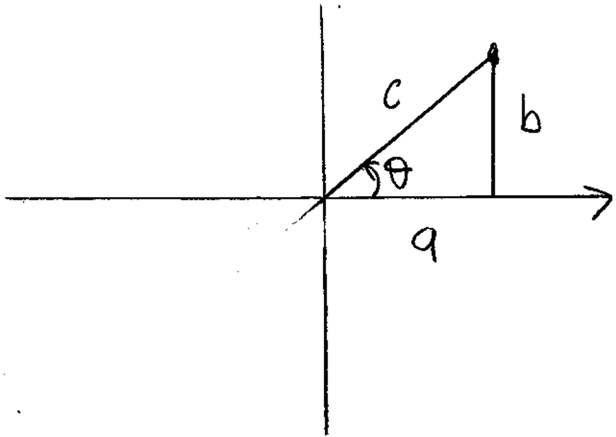
4. Find tan x if the terminal side of x is located at (4,3)

5. Find $W(\frac{\pi}{4})$ and Find θ if $W(\theta) = (-1/\sqrt{2}, 1/\sqrt{2})$

6. Using your calculator, compute sin (0.4) , sin (0.8)

5.3 Solving Right Triangles

Trigonometric Ratio

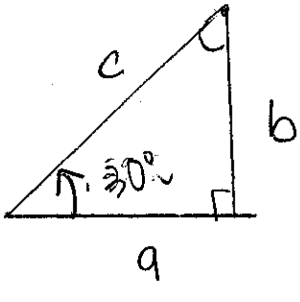


$\sin\theta = \frac{b}{c}$ $\cos\theta = \frac{a}{c}$, $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{b}{c} \times \frac{c}{a} = \frac{b}{a}$

$\cot\theta = \frac{a}{b}$, $\sec\theta = \frac{c}{a}$, $\csc\theta = \frac{c}{b}$

Example Solve the right triangle $c = 6$, $\beta = 30$

$\alpha = 90° - 30° = 60$, thus



$\cos 30° = \frac{b}{c} \Rightarrow \frac{\sqrt{3}}{2} = \frac{b}{c}$

$\Rightarrow b = \frac{\sqrt{3}c}{2} = \frac{6\sqrt{3}}{2} = 3\sqrt{3}$

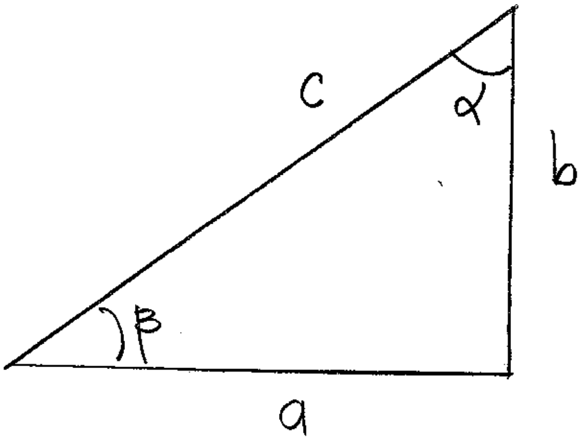
$\sin 30° = \frac{a}{c} \Rightarrow \frac{a}{c} = \frac{1}{2}$

$\Rightarrow a = \frac{c}{2} = \frac{6}{2} = 3$

$a = 3$, $b = 3\sqrt{3}$, $c = 6$.

Example

Given $\beta = 17.8°$, $c = 3.45$, Find the rest

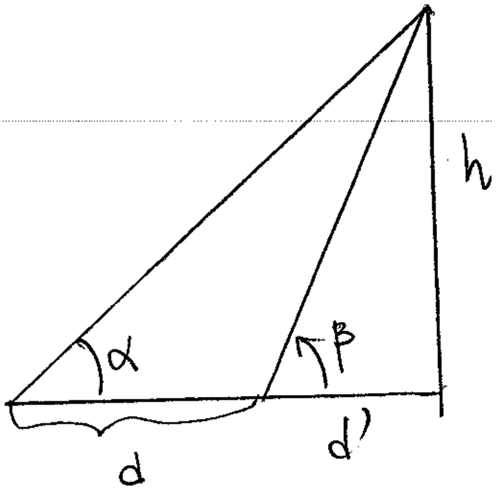


We know, $\frac{a}{c} = \cos\beta \Rightarrow a = c\cos\beta = 3.45\cos(17.8) = 3.28$

To go on degree mode hit "MODE" and change to degree.
Next, hit 2nd MODE to quit.

Next, $\frac{b}{c} = \sin\beta \Rightarrow b = c\sin\beta = 3.45\sin(17.8) = 1.05$

Example



Show $h = \dfrac{d}{\cot\alpha - \cot\beta}$

$\dfrac{h}{d'} = \tan\beta$, $\dfrac{h}{d+d'} = \tan\alpha$

$d' = h \cdot \text{cotan}\,\beta$, $h = (d+d')\tan\alpha$

thus $h = (d + h\,\text{cotan}\,\beta)\tan\alpha \Rightarrow h - h\,\text{cotan}\beta\tan\alpha = d\tan\alpha$

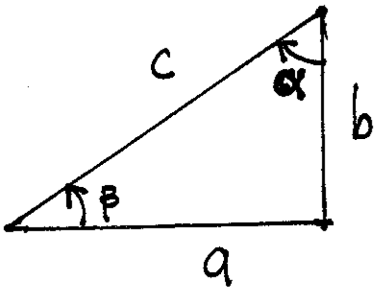
$\Rightarrow h(1 - \text{cotan}\,\beta\tan\alpha) = d$

$\Rightarrow h = \dfrac{d\tan\alpha}{1 - \text{cotan}\,\beta\tan\alpha} = \dfrac{d}{\text{cotan}\,\alpha - \text{cotan}\,\beta}$

$1 - \text{cotan}\,\beta$

Worsheet53

1.



Given $\alpha = 53.21$, $b = 23.82$, Find the rest

2. If α, β are the acute angles of a right triangle, then

$$\tan\alpha = \cot\beta$$

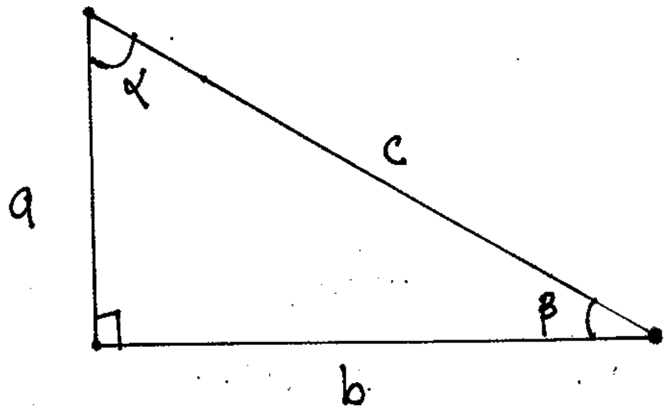
3. Find the slope of the Line passing through the ray which angle is a) $\theta = 30°$, b) $\theta = \pi/20$

<u>Hint</u> Solution for a.) 0.11, b.) 0.16.

Example

If α, β are the acute angles of a right triangle then

a) $\sec\alpha = \sec\beta$.



$\sec\alpha = \frac{1}{\cos\alpha}$. but $\cos\alpha = \frac{a}{c} \Rightarrow \sec\alpha = \frac{c}{a}$

$\sec\beta = \frac{1}{\cos\beta}$. But $\cos\beta = \frac{b}{c} \Rightarrow \sec\beta = \frac{c}{b}$ No.

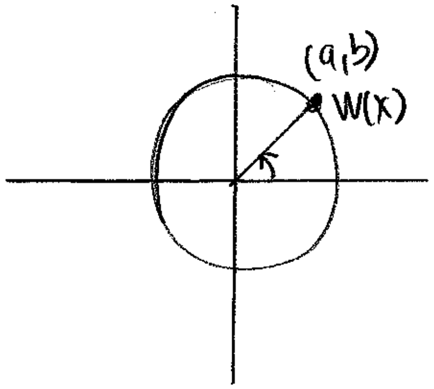
b) Is $\csc\alpha = \sec\beta$?

$\csc\alpha = \frac{1}{\sin\alpha}$, $\sin\alpha = \frac{b}{c} \Rightarrow \csc\alpha = \frac{c}{b} = \sec\beta$.

The answer is yes!

5.4 Properties of Trig functions

Definition Let $x \in \mathbb{R}$, $W(x) = (a,b)$



Remark1 If $W(x) = (a,b)$ then $W(-x) = (a,-b)$

and $W(x) = (\cos x, \sin x)$, $W(-x) = (\cos -x, \sin(-x))$

$\sin x = b$, $\sin(-x) = -b \Rightarrow \sin(-x) = -\sin x$.

Also, since, a, b belong to C, then $(\cos x)^2 + (\sin x)^2 = 1$

Remark 2 $(\sin x)^2 \neq \sin(x^2)$

$(\cos x)^2 \neq \cos x^2$

Basic Trig Identity

$\csc x = \frac{1}{\sin x}$, $\sec x = \frac{1}{\cos x}$, $\cot x = \frac{1}{\tan x}$

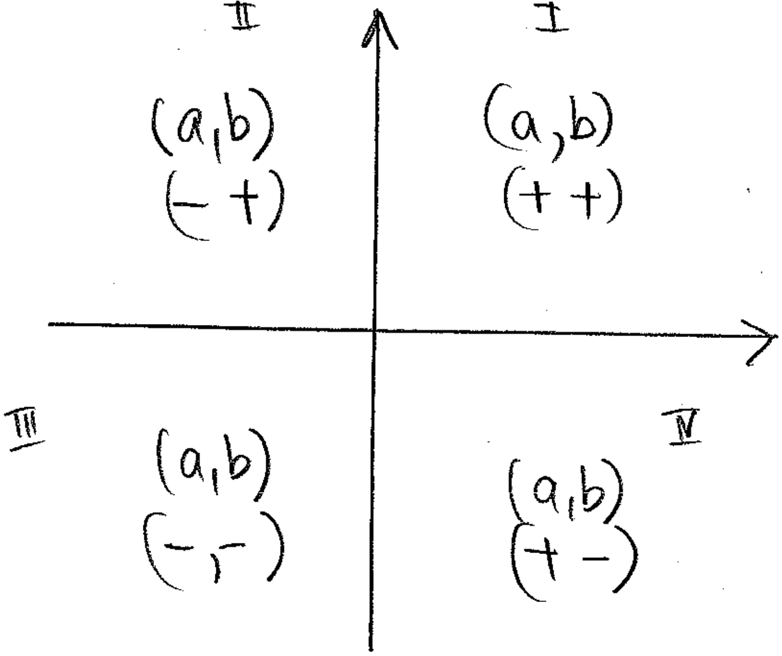
$\tan x = \frac{\sin x}{\cos x}$, $\cot x = \frac{\cos x}{\sin x}$

$\sin(-x) = -\sin x$, $\cos(-x) = \cos x$, $\tan(-x) = -\tan x$

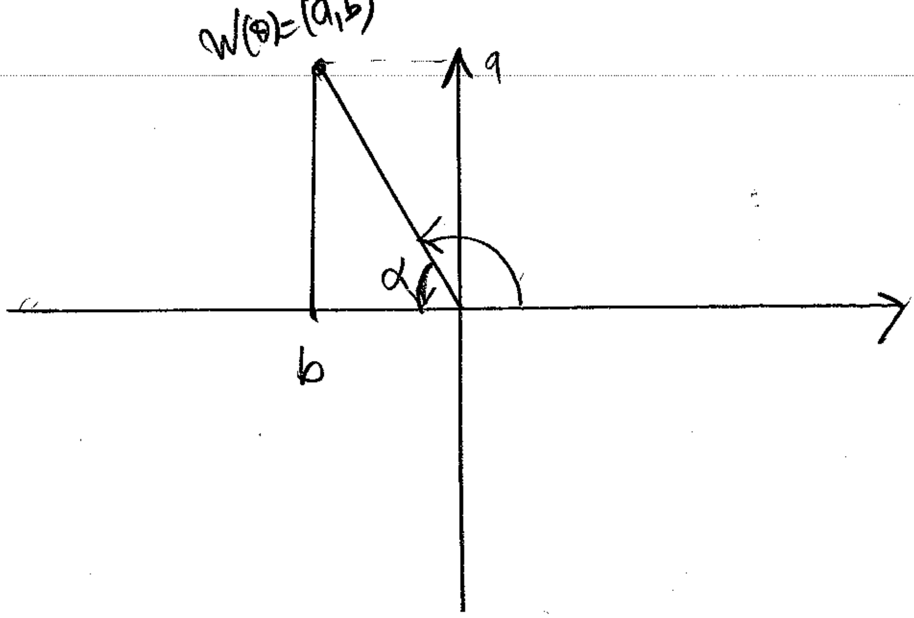
$\sin^2 x + \cos^2 x = 1$ (Pythagorean identity)

Sign properties

Let $W(x) = (a,b)$



Reference triangle and angle



For the reference triangle we use the acute angle the reference angle is always taken to be positive between the terminal side and the horizontal axis.

Example If $\sin\theta = 4/7$, $\cos\theta < 0$, find $\cos\theta$, $\tan\theta$, $\csc\theta$, $\sec\theta$, $\cot\theta$, assume θ is in the second quadrant.

$$\cos\theta^2 + \sin^2\theta = 1 \Rightarrow \cos^2\theta = 1 - \sin^2\theta$$

$$\Rightarrow \cos\theta = \pm\sqrt{1-\sin^2\theta}$$

$$\Rightarrow \cos\theta = \pm\sqrt{1 - \frac{16}{49}} = \sqrt{\frac{49-16}{49}}$$

$$\Rightarrow \cos\theta = \pm\frac{\sqrt{33}}{7}$$

thus $\cos\theta = -\frac{\sqrt{33}}{7}$

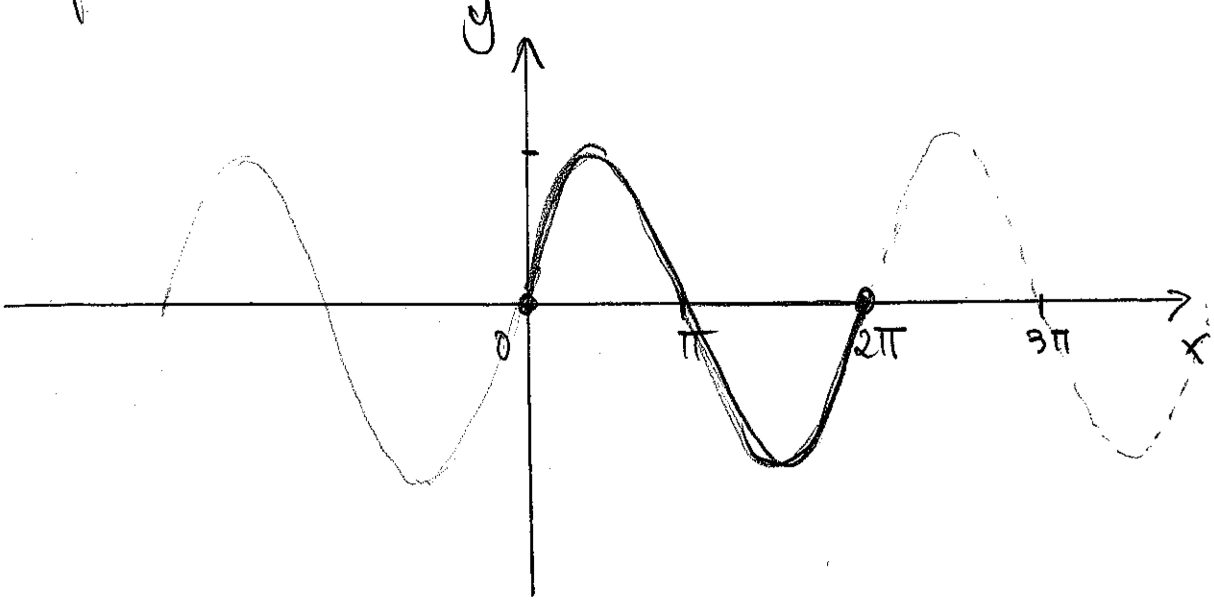
$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{4/7}{-\sqrt{33}/7} = \frac{4}{7} \times \frac{-7}{\sqrt{33}} = -\frac{4}{\sqrt{33}}$$

$$\tan\theta = \frac{-4\sqrt{33}}{33}.$$

Periodic Functions

Definition A function f is periodic if $\exists\, p \in \mathbb{R}$, $p > 0$ such that $f(x+p) = f(x)$. $\forall\, x \in D_f$

The smallest p if defined is called the fundamental period of f. or just period.



Symmetry

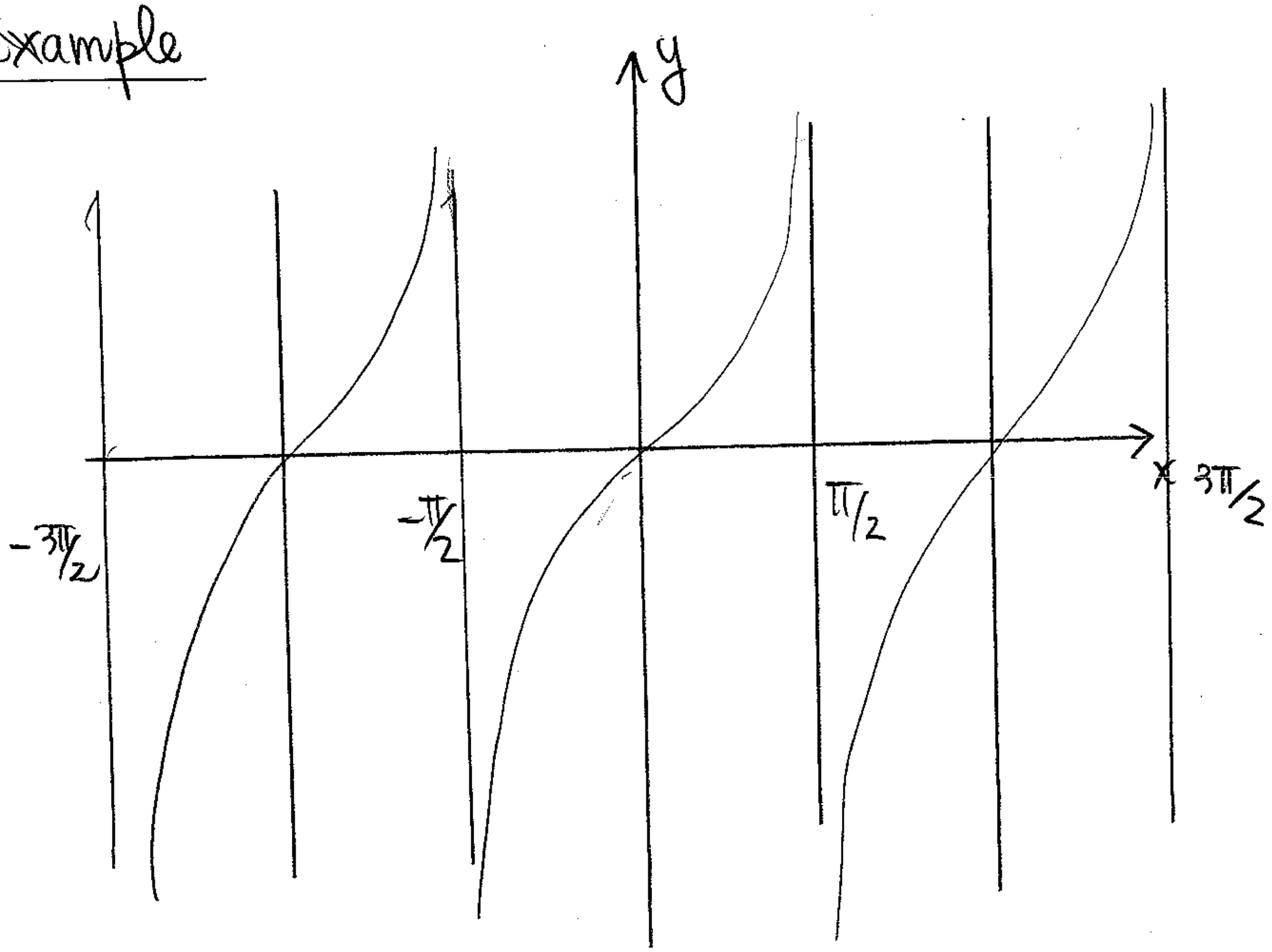
Question is $f(x) = \frac{\cos x}{x^2}$ even, odd or neither

$$f(-x) = \frac{\cos(-x)}{(-x)^2} = \frac{\cos x}{x^2} = f(x)$$

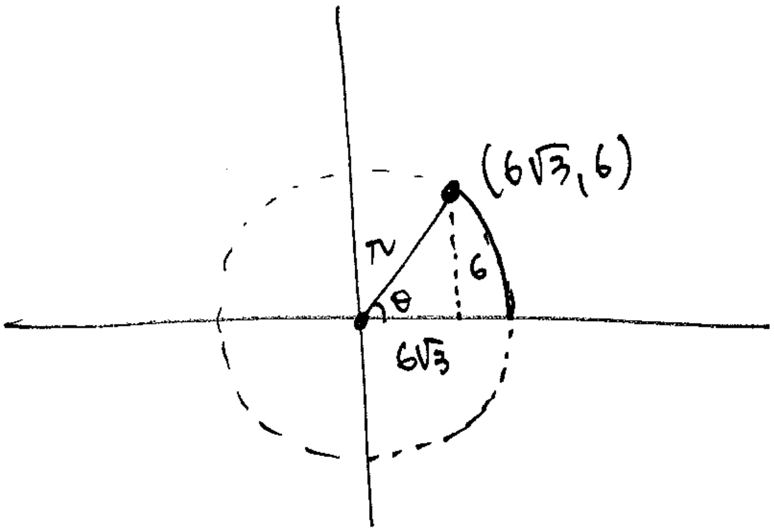
Thus f is even. Check that $f(x) = \frac{\sin x}{x}$ even also.
and that $g(x) = \frac{\cos x}{x}$ is odd.

Remark An even function is symmetrical with respect to the x-axis

Example



Example A circle with center at the origin passes through the pt $(6\sqrt{3}, 6)$. What is the length of the arc on the circle in quadrant I between the positive horizontal axis and the point $(6\sqrt{3}, 6)$.



$r^2 = (6\sqrt{3})^2 + 6^2 = 3 \times 36 + 36 = 144 \Rightarrow r = 12.$

We know that $s = r\theta$. Next, since $\sin\theta = \frac{6}{12} = \frac{1}{2} \Rightarrow \theta = 30^\circ$

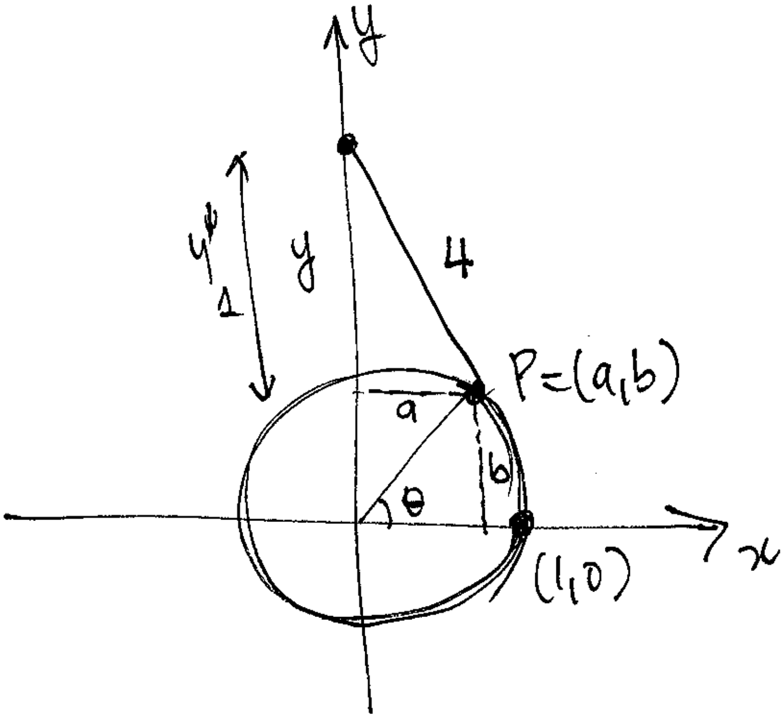
or $\theta = \frac{\pi}{6} \Rightarrow s = 12 \times \frac{\pi}{6} = 2\pi$.

Application

Example A piston connected to a wheel turns 3 revolutions per sec. The angle being generated is 6π rad / sec.

If P is at (1,0) when $t=0$ show that $y = b+\sqrt{4^2-a^2}$

$= \sin 6\pi t + \sqrt{16-(\cos 6\pi t)^2}$



angular speed $= \frac{\theta}{t} = 6\pi / s \Rightarrow \theta = 6\pi t.$

$y = y_1 + b$. But $y_1^2 + a^2 = 4^2 \Rightarrow y_1^2 = 4^2 - a^2 \Rightarrow y_1 = \sqrt{4^2-a^2}$

Thus $y = b+\sqrt{4^2-a^2}$.

5.5 – More general Trig functions and Models

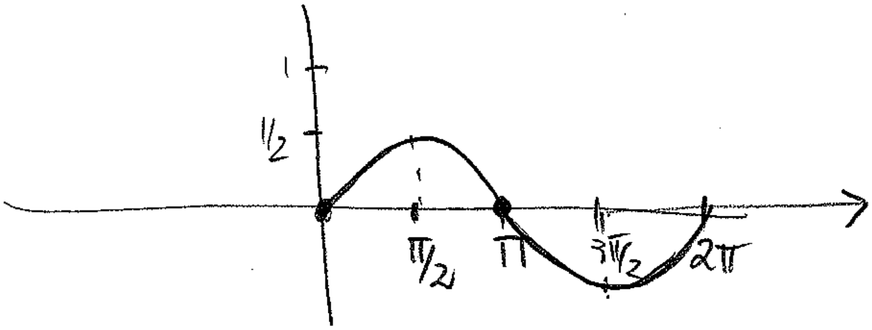
A motion of a free weight moving without air friction is called a simple harmonic motion.

A simple harmonic function is of the form

$$f(x) = A\sin(Bx+C) \text{ or } f(x) = A\cos(Bx+c)$$

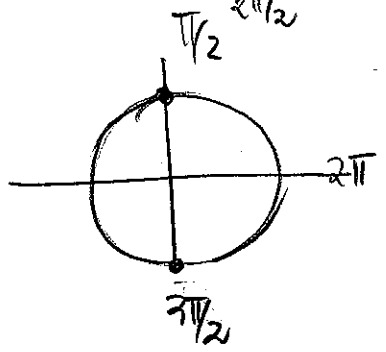
Example Find the zeros and turning points for $f(x) = \frac{1}{2}\sin x$ on the interval $[0, 2\pi]$.

$$f(x) = 0 \Rightarrow \sin x = 0 \Rightarrow x = 0, \pi, 2\pi$$

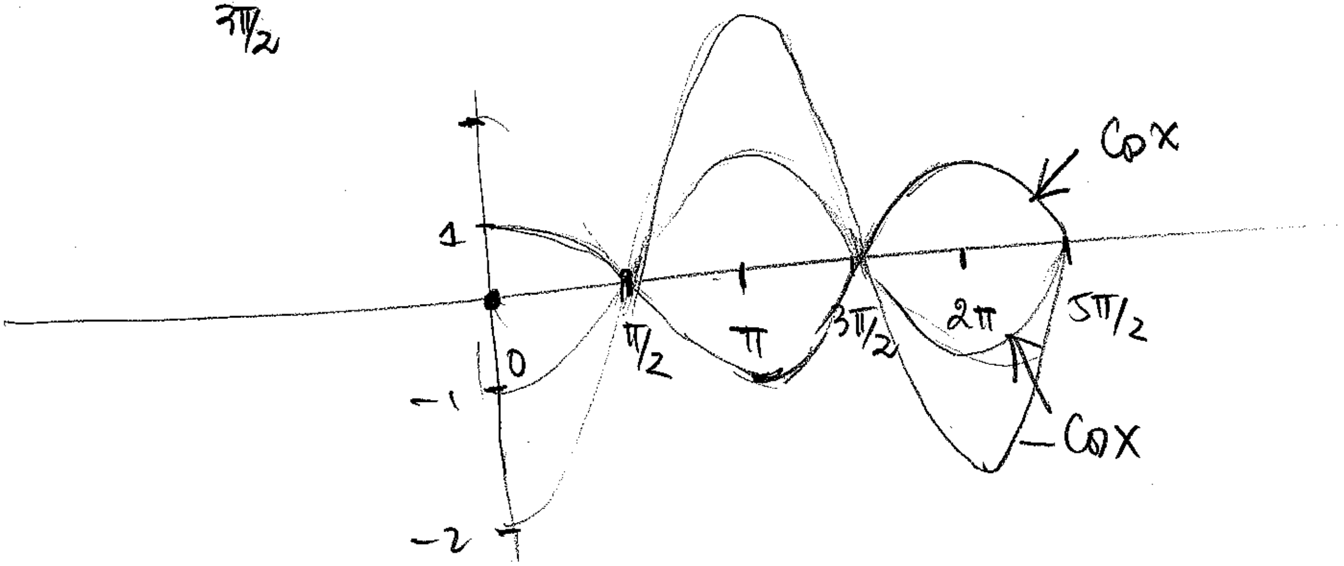


Turning points: $(\pi/2, 1/2)$ and $(3\pi/2, -1/2)$

Example: Find the zeros and turning points for $f(x) = -2\cos x$ on $[\pi/2, 5\pi/2]$

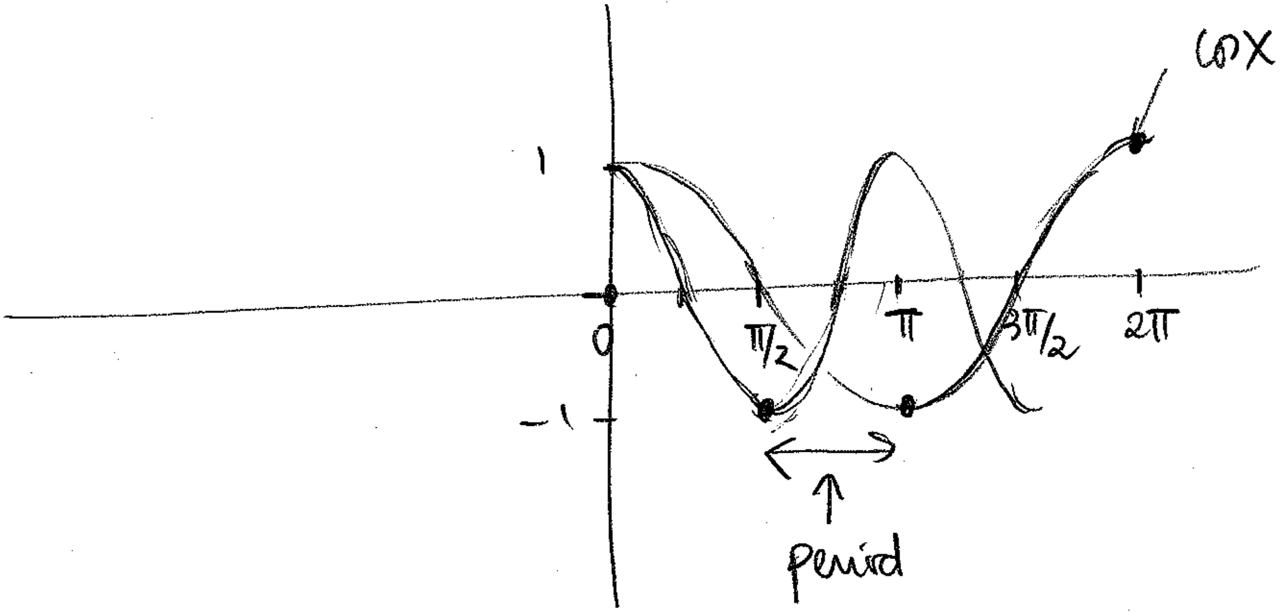


$x = \pi/2$, $x = 3\pi/2$, $x = \frac{5\pi}{2}$



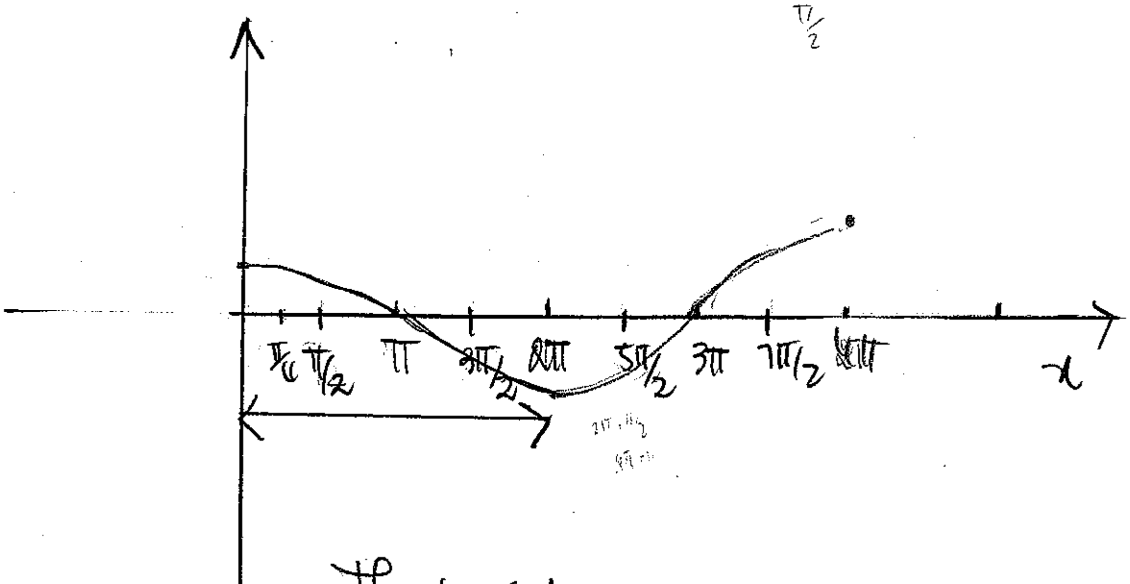
The turning points are: $(\pi, 2)$ and $(2\pi, -2)$

Example Find the period for $\cos 2x$



The period of $\cos 2x$ is π

Example Find the period for $\cos(\frac{1}{2}x)$



The period is 4π

Find the period of $\cos(6\pi x)$: $6\pi(2\pi) = 12\pi^2$

Period and Amplitude

For $f(x) = A\sin Bx$ or $f(x) = A\cos Bx$, $A \neq 0$, $B > 0$

The amplitude $= |A|$, period $= \dfrac{2\pi}{B}$

Example Find the amplitude, period and turning points of $y = -3\cos\left(\dfrac{\pi x}{2}\right)$ on the interval $[-\pi, \pi]$.

The amplitude is $|-3| = 3$.

$$\text{period} = \frac{2\pi}{B} = \frac{2\pi}{\frac{1}{2}\pi} = 4$$

Graphs of $y = A\sin(Ax+C)$ or $A\cos(Ax+C)$

Given $f(x) = A\sin(Bx+c)$; $g(x) = A\sin(Bx)$

1. Notice that f and g have the same period; since they are horizontal translate of each other
2. Thus f has for period $p = \dfrac{2\pi}{B}$
3. The phase shift is $-\dfrac{C}{B}$

For 1: $f\left(x - \frac{c}{B}\right) = A\sin\left(B\left(x - \frac{c}{B}\right) + c\right)$

$= A\sin(Bx)$

Example

Given $y=\frac{1}{2}\cos(4x-\pi)$

$A=\frac{1}{2}, B=4, C=-\pi$

(A) the amplitude is $\frac{1}{2}$

(B) Finding all zeros

$$y=0 \Rightarrow 4x-\pi=\frac{\pi}{2}+2k\pi \Rightarrow 4x=\pi+\frac{\pi}{2}+2k\pi, k\in\mathbb{Z}$$

$$\Rightarrow 4x=\frac{3\pi}{2}+2k\pi \Rightarrow x=\frac{3\pi}{8}+\frac{1}{2}k\pi, k\in\mathbb{Z}$$

$$x=\frac{3\pi}{8}, \frac{3\pi}{8}+\frac{\pi}{2}, \frac{3\pi}{8}+\pi, \dots$$

(D) Find the smallest positive value such that

$$\frac{1}{2}\cos(4x-\pi)=-\frac{1}{2} \Rightarrow \cos(4x-\pi)=-1$$

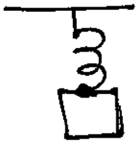
$$\Rightarrow \cos(4x-\pi)=\cos(\pi)$$

$$\Rightarrow 4x-\pi=\pi+2k\pi$$

$$\Rightarrow 4x=2\pi+2k\pi$$

$$\Rightarrow x=\frac{\pi+k\pi}{2}=\frac{\pi(k+1)}{2}$$

$$\Rightarrow x=\frac{(k+1)\pi}{2}, k\in\mathbb{Z}$$



Application

Harmonic Motion A weight on the end of a spring is oscillating in harmonic motion.

The equation model is $d(t) = 6\sin(\frac{\pi}{2}t)$, where d is the distance from the equilibrium point in t sec.

a.) What is the period of the motion?

$$d(t) = A\sin(Bt+c), \quad A = 6, \quad B = \frac{\pi}{2}, \quad c = 0$$

The period is $\frac{2\pi}{B} = \frac{2\pi}{\frac{\pi}{2}} = 2\pi \times \frac{2}{\pi} = 4$

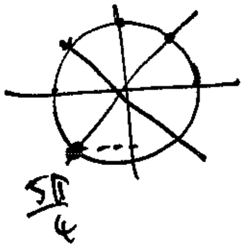
b.) What is the frequency of the of the motion?

$f = \frac{1}{4}$, $\left(\text{frequency} = \frac{1}{\text{period}}\right)$.

c.) What is the displacement from equilibrium at $t = 2.5$.

$$d(2.5) = 6\sin\left(\frac{2.5\pi}{2}\right) = 6\sin\left(\frac{5\pi}{4}\right)$$

$$= -6\times\frac{\sqrt{2}}{2} = -3\sqrt{2}$$



displacement $= 3\sqrt{2}$.

5.6 Inverse Trig functions

Some facts about Inverses

- Domain of f = Range of f^{-1}
- Range of f = Domain of f^{-1}
- If $y = f(x) \Rightarrow x = f^{-1}(y)$
- $f^{-1}(f(x)) = x$ and $f(f^{-1}(y)) = y$
- Domain of $f \circ f^{-1}$ = Domain of f^{-1}
- Range of $f^{-1} \circ f$ = Domain of f

Remark f is not injective on $\mathbb{R}$ but restricted to $[-\pi/2, \pi/2]$, f is injective.

Definition $\arcsin(x) = \sin^{-1}(x)$

ie if $\arcsin(x) = y \Rightarrow \sin(y) = x \quad \forall \ x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

Properties

1. $\sin(\arcsin x) = x$, $x \in [-1, 1]$
2. $\arcsin(\sin \theta) = \theta$, $\forall \ \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

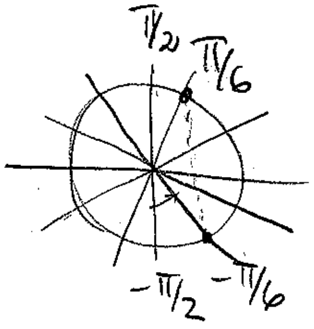
Definition $\arccos(x) = \cos^{-1}(x)$ Defined on $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

1. $\cos(\arccos x) = x$, $x \in [-1, 1]$

~~2. $\sin(\arcsin x) =$~~

2. $\arccos(\cos \theta) = \theta$, $\forall \ \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

Examples Without a calculator, compute



(A) $\arcsin\left(-\frac{1}{2}\right)$

Let $y = \arcsin\left(-\frac{1}{2}\right) \Rightarrow \sin(y) = -\frac{1}{2}$

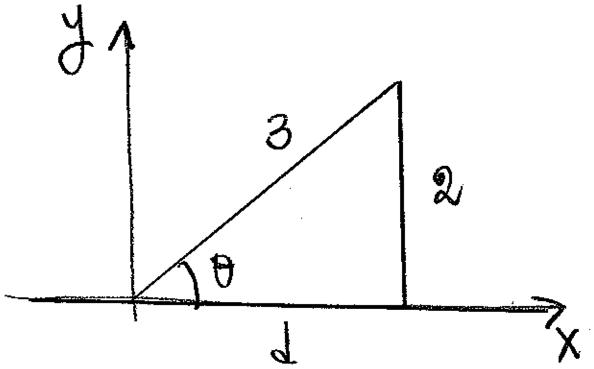
We know that $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$

$$\arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$$

(B) $\sin^{-1}(\sin(1.2)) = 1.2$

(C) $\cos\left(\sin^{-1}\left(\frac{2}{3}\right)\right)$

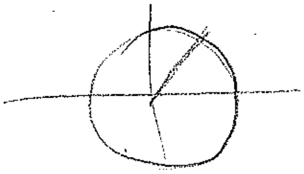
Put $y = \arcsin\left(\frac{2}{3}\right) \Rightarrow \sin y = \frac{2}{3}$



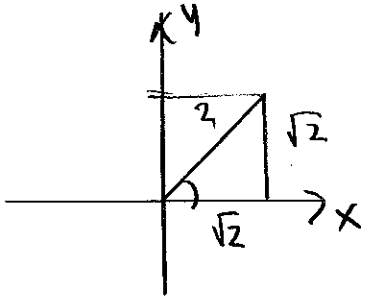
$d^2 + 4 = 9 \Rightarrow d = \sqrt{5} \Rightarrow \cos\theta = \frac{d}{3} = \frac{\sqrt{5}}{3}$

Example: Without a calculator, compute

(A) $\arcsin(\sqrt{2}/2)$



Put $y = \arcsin \frac{\sqrt{2}}{2} \Rightarrow \sin y = \frac{\sqrt{2}}{2}$



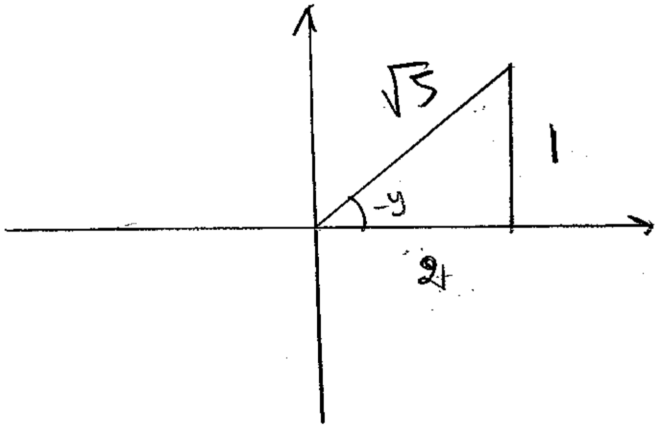
Thus $y = \frac{\pi}{4}$

(B) $\tan\left(\sin^{-1}\left(-\frac{1}{\sqrt{5}}\right)\right)$.

Put $y = \sin^{-1}\left(-\frac{1}{\sqrt{5}}\right)$

$\sin y = -\frac{1}{\sqrt{5}}$

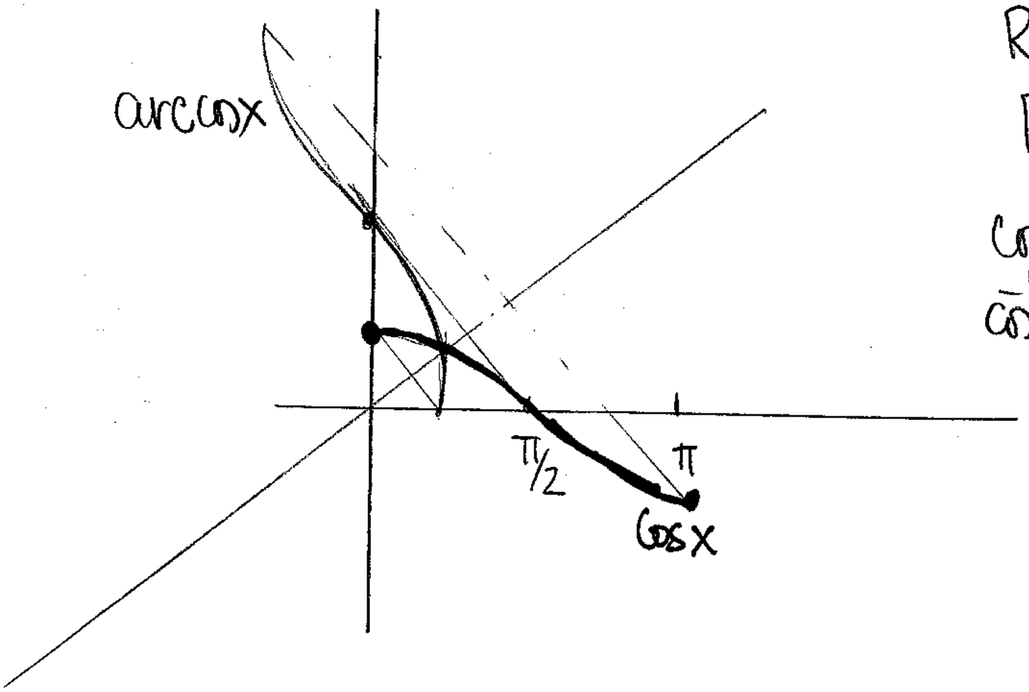
$\sin(-y) = \frac{1}{\sqrt{5}}$



$\tan\left(\sin^{-1}\left(-\frac{1}{\sqrt{5}}\right)\right)$

$= \frac{1}{2}$.

Inverse cosine function

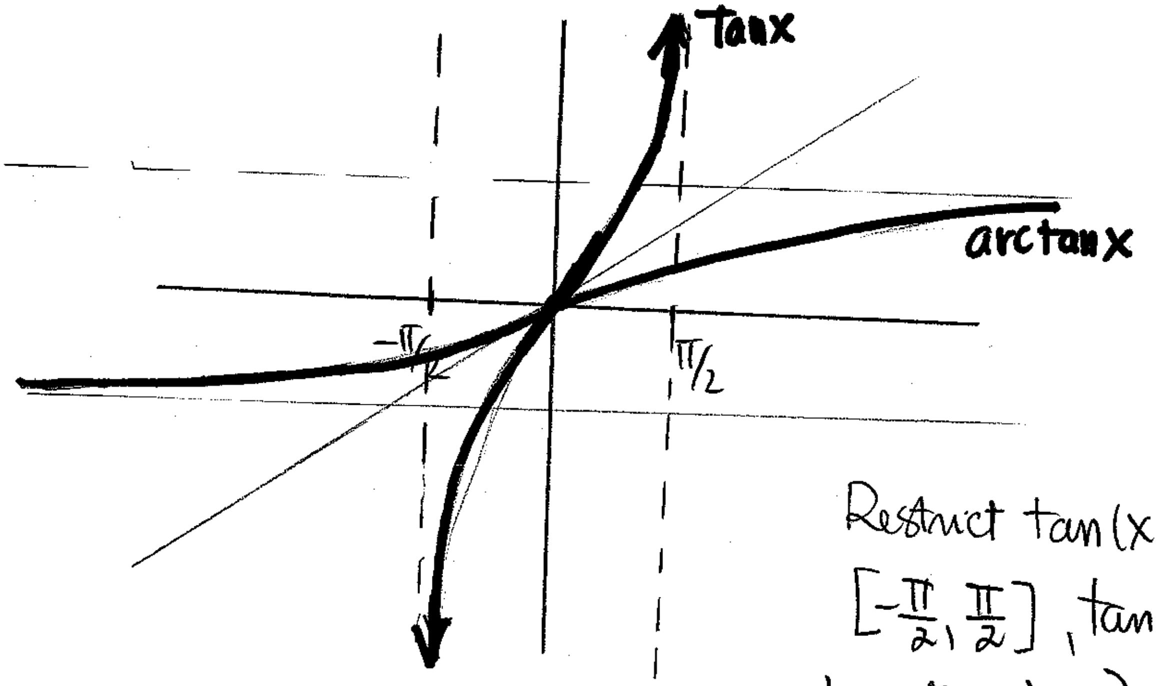


Restrict cos(x) to
[0, 2π]

$\cos(\cos^{-1}(x)) = x, \ x \in [-1, 1]$

$\cos^{-1}(\cos x) = x, \ x \in [0, 2\pi]$

Inverse tangent function



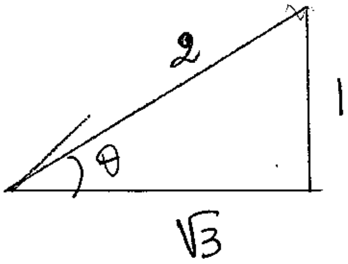
Restrict tan(x) to
$[-\frac{\pi}{2}, \frac{\pi}{2}]$, tan is one-to-one

$\tan(\tan^{-1}(x)) = x, \ x \in \mathbb{R}$

$\tan^{-1}(\tan x) = x, \ x \in [\frac{-\pi}{2}, \frac{\pi}{2}]$

Example

(A) $\arctan\left(\frac{1}{\sqrt{3}}\right)$, $y = \arctan\left(\frac{1}{\sqrt{3}}\right) \Rightarrow \tan y = \frac{1}{\sqrt{3}} = \frac{opp}{adj}$, $y \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$



, Clearly $\sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}$

Thus $\arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$

(B) $\tan^{-1}(-1)$ Put $y = \tan^{-1}(-1)$, $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Thus, $\tan y = -1 \Rightarrow \tan(-y) = 1$

$$\Rightarrow -y = \frac{\pi}{4}$$

$$\Rightarrow y = -\frac{\pi}{4}.$$

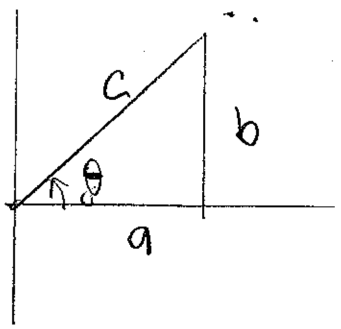
Example Express as an algebraic expression

(A) $\cos(\sin^{-1}(x)) = y$

$$\cos(\sin^{-1}(x))^2 + \sin(\sin^{-1}(x))^2 = 1$$

$$\cos(\sin^{-1}(x))^2 + x^2 = 1$$

$$\cos(\sin^{-1}(x)) = \sqrt{1-x^2}$$



(B) $\cos(\arctan x) = y \Rightarrow$

$$\frac{\sin(\arctan x)}{\cos(\arctan x)} = \tan(\arctan x) = x$$

But $$\frac{\sin^2(\arctan x) + \cos^2(\arctan x)}{\cos^2(\arctan x)} = \frac{1}{y^2}$$

$$x^2 + 1 = \frac{1}{y^2} \Rightarrow y^2 = \frac{1}{x^2+1}$$

(c)

$y = \tan(\arcsin x)$ get rid of tan

$$\tan(\arcsin x)\cos(\arcsin x) = y\cos(\arcsin x)$$

$$\sin(\arcsin x) = y\cos(\arcsin x)$$

$$x = y\cos(\arcsin x)$$

thus $y = \dfrac{x}{\cos(\arcsin x)} = \dfrac{x}{\sqrt{1-x^2}}$.

6.1 Basic Trig Identity

$\sin(-x) = -\sin(x)$
$\cos(-x) = \cos(x)$
$\tan(-x) = -\tan(x)$

$\sin^2(x) + \cos^2(x) = 1$
$\tan^2(x) + 1 = \sec^2(x)$
$1 + \cot an^2(x) = \csc^2(x)$

Example 1: Prove that

$$\frac{\cos(x) - \sin(x)}{\sin x \cos(x)} = \csc x - \sec x$$

$$\frac{1}{\sin x} - \frac{1}{\cos(x)} = \csc x - \sec x$$

Example 2: $\sin\theta + \cos\theta = \dfrac{\tan\theta + 1}{\sec\theta}$

$$\frac{\tan\theta + 1}{\sec\theta} = \frac{\tan\theta}{\sec\theta} + \frac{1}{\frac{1}{\cos\theta}}$$

$$= \frac{\sin\theta}{\cos\theta} \times \frac{1}{\sec\theta} + \cos\theta$$

$$= \sin\theta + \cos\theta$$

Example 3 $\ln(\tan(x)) = \ln(\sin x) - \ln(\cos(x))$

Example 4 $\sin^4 w - \cos^4 w = 1 - 2\cos^2 w$

$$(\sin^2 w - \cos^2 w)(\sin^2 w + \cos^2 w)$$

$$= 1(\sin^2 w - \cos^2 w)$$

$$= 1 - \cos^2 w - \cos^2 w = 1 - 2\cos^2 w$$

Example 5 $\frac{\tan u + \sin u}{\tan u - \sin u} - \frac{\sec u + 1}{\sec u - 1} = 0$

$$\frac{\tan u + \sin u}{\tan u - \sin u} = \frac{\frac{\sin u}{\cos u} + \sin u}{\frac{\sin u}{\cos u} - \sin u}$$

$$= \frac{\sin u + \sin u \cos u}{\cos u} * \frac{\cos u}{\sin u - \sin u \cos u}$$

$$= \frac{\sin u + \sin u \cos u}{\sin u - \sin u \cos u} = \frac{\sin u(1 + \cos u)}{\sin u(1 - \cos u)}$$

$$= \frac{1 + \cos u}{1 - \sin u} = \frac{1 + \frac{1}{\sec u}}{1 - \frac{1}{\sec u}}$$

$$= \frac{\frac{\sec u + 1}{\sec u}}{\frac{\sec u - 1}{\sec u}} = \frac{\sec u + 1}{\sec u - 1}.$$

Example 6

$$\tan\alpha + \cot\beta = \frac{\tan\beta + \cot\alpha}{\tan\beta + \cot\alpha}$$

$$\frac{1}{\cot\alpha} + \frac{1}{\tan\beta} = \frac{\tan\beta + \cot\alpha}{\cot\alpha + \tan\beta}$$

Ex1 Verify

$$\frac{\sin^6 x - \cos^6 x}{\sin^4 x - \cos^4 x} = 1 - \sin^2 x \cos^2 x$$

$$\frac{(\cancel{\sin^3 x - \cos^3 x})(\cancel{\sin^3 x + \cos^3 x})}{(\sin^2 x - \cos^2 x)(\cancel{\sin^2 x + \cos^2 x})} \qquad \cancel{a^3 - b^3 = (a-b)(a^2+ab+b^2)}$$

$$= \frac{\sin^6 x - \cos^6 x}{\sin^2 x - \cos^2 x} = \frac{(\sin^2 x)^3 - (\cos^2 x)^3}{\sin^2 x - \cos^2 x}$$

$$= \frac{(\cancel{\sin^2 x - \cos^2 x})(\sin^4 x + \sin^2 x \cos^2 x + \cos^4 x)}{\cancel{\sin^2 x - \cos^2 x}}$$

$$= \sin^4 x + \sin^2 x \cos^2 x + \cos^4 x$$

$$= \sin^4 x + 2\sin^2 x \cos^2 x + \cos^4 x - \sin^2 x \cos^2 x$$

$$= (\sin^2 x + \cos^2 x)^2 - \sin^2 x \cos^2 x$$

$$= 1 - \sin^2 x \cos^2 x.$$

Ex2

$$\frac{\cot x - \tan x}{\cot^2 x - \tan^2 x} = \frac{\cot x - \tan x}{(\cot x - \tan x)(\cot x + \tan x)}$$

$$= \frac{1}{\cot x + \tan x} = \frac{1}{\frac{\cos x}{\sin x} + \frac{\sin x}{\cos x}} = \frac{1}{\frac{\cos^2 x + \sin^2 x}{\sin x \cos x}}$$

$$= \sin x \cos x .$$

Ex3 Show $$\frac{\csc x}{\cot x + \tan x} = \cos x$$

$$\frac{\frac{1}{\sin x}}{\frac{\cos x}{\sin x} + \frac{\sin x}{\cos x}} = \frac{\frac{1}{\sin x}}{\frac{\cos^2 x + \sin^2 x}{\sin x \cos x}}$$

$$= \frac{1}{\cancel{\sin x}} \times \frac{\cancel{\sin x} \cos x}{1} = \cos x .$$

6.2 - Sum, Difference and Cofunction Identities

Sum Identities

$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$
$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$
$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

Difference Identities

$$\sin(x-y) = \sin x \cos y - \cos x \sin y$$
$$\cos(x-y) = \cos x \cos y + \sin x \sin y$$
$$\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

Cofunction Identities

$$\cos\left(\frac{\pi}{2} - x\right) = \sin x$$
$$\sin\left(\frac{\pi}{2} - x\right) = \cos x$$
$$\tan\left(\frac{\pi}{2} - x\right) = \cot x$$

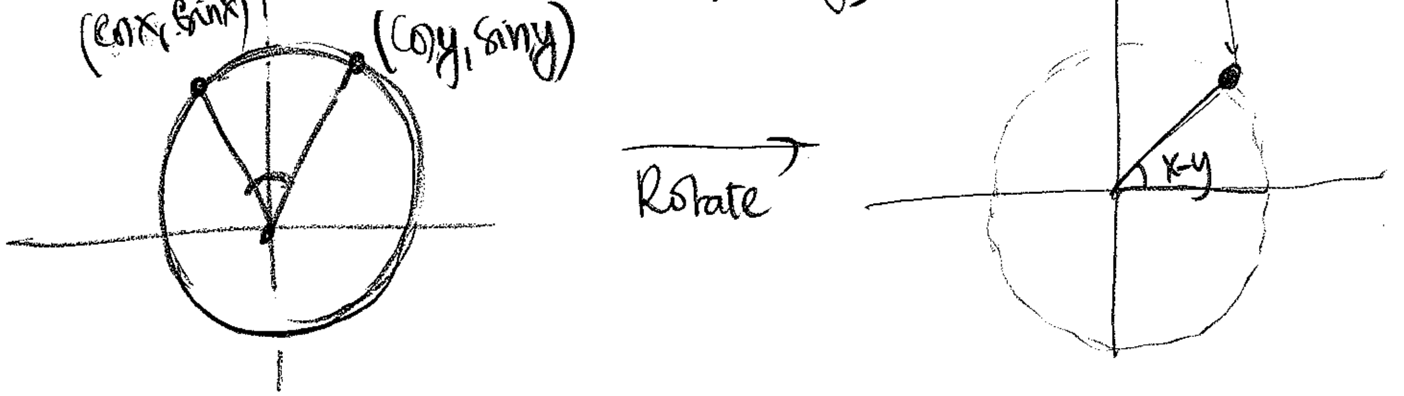
6.2 Sum, Difference and Cofunction Identities

Difference Identity for cosine

Sketch of proof

- $\cos(x-y) = \cos x \cos y + \sin x \sin y$

$$\begin{bmatrix} \cos y & \sin y \\ \sin y & -\cos y \end{bmatrix}$$



Thus $\cos(x-y) = \cos x \cos y + \sin x \sin y$

- $\cos\left(\frac{\pi}{2}-y\right) = \sin y$ (Cofunction Identity for cosine)

$$\cos\left(\frac{\pi}{2}-y\right) = \cos\frac{\pi}{2}\cos y + \sin\frac{\pi}{2}\sin y$$
$$= \sin y$$

- $\sin\left(\frac{\pi}{2}-x\right) = \cos x$ (Cofunction identity for sine)

$$\cos(x) = \cos\left(\frac{\pi}{2}-\frac{\pi}{2}+x\right) = \cos\left(\frac{\pi}{2}-\left(\frac{\pi}{2}-x\right)\right)$$
$$= \sin\left(\frac{\pi}{2}-x\right) \text{ by above.}$$

Difference Identity for sine

$$\sin(x-y) = \sin x \cos y - \cos x \sin y$$

$$\sin(x-y) = \cos\left(\frac{\pi}{2} - (x-y)\right)$$

$$= \cos\left(\left(\frac{\pi}{2} - x\right) - (-y)\right)$$

$$= \cos\left(\frac{\pi}{2} - x\right)\cos(-y) + \sin\left(\frac{\pi}{2} - x\right)\sin(-y)$$

$$= \sin x \cos y - \cos x \sin y.$$

Example 1 Simplify $\sin(x-\pi)$

$$\sin x \cos\pi - \cos x \sin\pi$$

$$= -\sin x.$$

Example Simplify $\cos\left(x+\frac{3\pi}{2}\right)$ using sum Identity

$$\cos x \cos\frac{3\pi}{2} - \sin x \sin\frac{3\pi}{2} = \sin x.$$

Example Find the exact value of $\cos 15°$ in radical form.

$$\begin{aligned}\cos(15°) &= \cos(60° - 45°)\\ &= \cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right)\\ &= \cos\frac{\pi}{3}\cos\frac{\pi}{4} + \sin\frac{\pi}{3}\sin\frac{\pi}{4}\\ &= \frac{1}{2}\frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2}\\ &= \frac{\sqrt{2}+\sqrt{6}}{4}.\end{aligned}$$

5+6

Example Find the exact value of tan $(11\pi/12)$ in Radical form.

$$\frac{11\pi}{12} = \frac{2\pi + 9\pi}{12} = \frac{\pi}{6} + \frac{3\pi}{4}$$

2π .

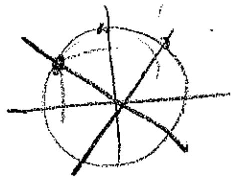
$$\tan\left(\frac{11\pi}{12}\right) = \tan\left(\frac{\pi}{6} + \frac{3\pi}{4}\right)$$

30

$$= \frac{\tan\frac{\pi}{6} + \tan\frac{3\pi}{4}}{1 - \tan\frac{\pi}{6}\tan\frac{3\pi}{4}}$$

$$\left\{ \begin{array}{l} \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} \\ \sin\frac{\pi}{6} = \frac{1}{2} \end{array} \right. \Rightarrow \tan\frac{\pi}{6} = \frac{1}{2} \times \frac{2}{\sqrt{3}} = \frac{1}{\sqrt{3}}$$

$\frac{180°}{6}$



$$= \frac{\frac{1}{\sqrt{3}} - 1}{1 + \frac{1}{\sqrt{3}}} = \frac{\frac{1-\sqrt{3}}{\sqrt{3}}}{\frac{\sqrt{3}+1}{\sqrt{3}}}$$

$$= \frac{1-\sqrt{3}}{\sqrt{3}+1}$$

6.3

Double-Angle and Half Angle Identities ③

Double-Angle Identities

1. $\sin 2x = 2\sin x \cos x$
2. $\cos 2x = \cos^2 x - \sin^2 x$
3. $\cos 2x = 1 - 2\sin^2 x$
4. $\cos 2x = 2\cos^2 x - 1$

Proof

1. $\sin(2x) = \sin(x+x) = \sin x \cos x + \sin x \cos x = 2\sin x \cos x$
2. $\cos 2x = \cos(x+x) = \cos x \cos x - \sin x \sin x = \cos^2 x - \sin^2 x$
3. $\cos 2x = 1 - \sin^2 x - \sin^2 x = 1 - 2\sin^2 x$
4. $\cos 2x = \cos^2 x - (1 - \cos^2 x) = 2\cos^2 x - 1$.

Half-Angle Identities

1. $\sin \frac{x}{2} = \pm \sqrt{\frac{1-\cos x}{2}}$
2. $\cos \frac{x}{2} = \pm \sqrt{\frac{1+\cos x}{2}}$
3. $\tan \frac{x}{2} = \pm \sqrt{\frac{1-\cos x}{1+\cos x}}$

$$= \frac{\sin x}{1+\cos x}$$

$$= \frac{1-\cos x}{\sin x}$$

Proof

1. We have $\cos 2x = 1 - 2\sin^2(x)$

$$\cos(x) = 1 - 2\sin^2\left(\frac{x}{2}\right)$$

Thus $2\sin^2\left(\frac{x}{2}\right) = 1 - \cos(x) \Rightarrow \sin\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1-\cos x}{2}}$

2. We have $\cos 2x = 2\cos^2 x - 1$. Thus

$$\cos(x) = 2\cos^2\left(\frac{x}{2}\right) - 1 \Rightarrow 2\cos^2\left(\frac{x}{2}\right) = \cos(x) + 1$$

$$\Rightarrow \cos^2\left(\frac{x}{2}\right) = \frac{\cos(x)+1}{2}$$

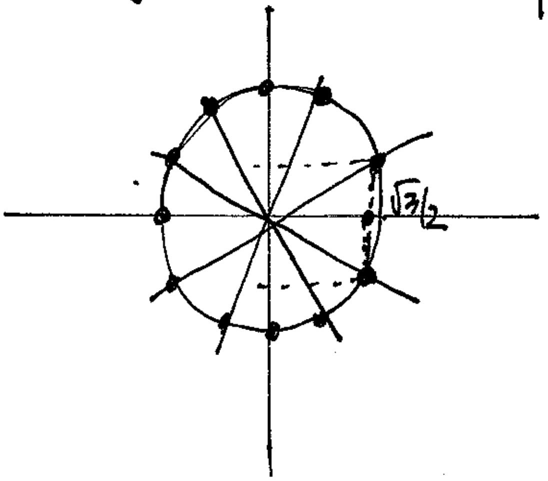
$$\Rightarrow \cos\left(\frac{x}{2}\right) = \pm\sqrt{\frac{\cos(x)+1}{2}}.$$

We omit the rest (Do for homework)

Examples Compute the exact value of $\sin 165°$

$$\begin{array}{l} \pi \to 180 \\ \theta \mapsto 165 \end{array} \Rightarrow \theta = \frac{165\pi}{180} = \frac{33\pi}{36} = \frac{11\pi}{12}$$

Thus $\sin(165°) = \sin\left(\frac{11\pi}{12}\right) = \sin\left(\frac{11\pi/6}{2}\right)$



$$\sin\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}}$$

$$\cos\left(\frac{11\pi}{6}\right) = \frac{\sqrt{3}}{2}\ , \quad \sin\frac{11\pi}{12} = \pm\sqrt{\frac{1-\cos\left(\frac{11\pi}{6}\right)}{2}}$$

$$= \pm\sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}}$$

$$= \pm\sqrt{\frac{\frac{2-\sqrt{3}}{2}}{2}}$$

$$= \pm\sqrt{\frac{2-\sqrt{3}}{4}}$$

$$= \pm\frac{\sqrt{2-\sqrt{3}}}{2}$$

Since $\frac{11\pi}{12}$ is in the 2nd quadrant, then

$$\sin\frac{x}{2} = \frac{\sqrt{2-\sqrt{3}}}{2}.$$

Example Find the exact value of cos 15°

$$\cos\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1+\cos x}{2}}$$

$$\begin{matrix} \pi \longrightarrow 180^\circ \\ \theta \longrightarrow 15^\circ \end{matrix} \Rightarrow \theta = \frac{15\pi}{180} = \frac{\pi}{12}.$$

$$\cos(15^\circ) = \cos\left(\frac{\pi}{12}\right) = \cos\left(\frac{\pi/6}{2}\right)$$

However $\cos \pi/6 = \frac{\sqrt{3}}{2}$. Since 15° is in the first quadrant, ~~it must~~ $\cos(15^\circ) \geq 0$.

Next $\cos(15^\circ) = \sqrt{\frac{1+\cos 30^\circ}{2}} = \sqrt{\frac{1+\sqrt{3}/2}{2}}$

$$= \frac{\sqrt{2+\sqrt{3}}}{2}$$

Example Compute without a calculator Sin (112.5°)

$$\begin{matrix} \pi \longrightarrow 180 \\ \theta \longleftrightarrow 112.5 \end{matrix} \Rightarrow \theta = \frac{112.5\pi}{180}$$

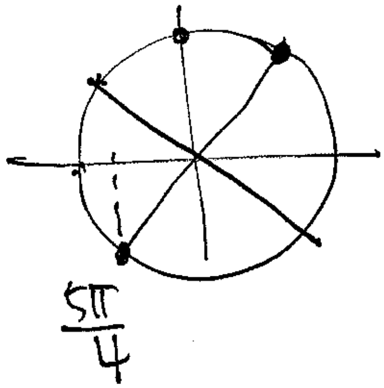
$$\theta = \frac{1125\pi}{1800} = \frac{225\pi}{360} = \frac{45\pi}{72} = \cancel{\frac{3\cdot 15\pi}{3\cdot 36\pi}}$$

$$= \cancel{\frac{5\pi}{18}} = \cancel{\frac{5\pi}{12}}.$$

$$= \frac{5\pi}{8}$$

$$\sin\left(\frac{5\pi}{8}\right) = \sin\left(\frac{5\pi/4}{2}\right) = \sqrt{\frac{1-\cos x}{2}}$$

$$= \sqrt{\frac{1-\cos 5\pi/4}{2}} = \sqrt{\frac{1+\frac{\sqrt{2}}{2}}{2}}$$



$$= \sqrt{\frac{2+\sqrt{2}}{4}} = \frac{\sqrt{2+\sqrt{2}}}{2}.$$

Exercise Verify that $\sin 2x = \tan x(1+\cos 2x)$

We have $\frac{\sin 2x}{1+\cos 2x} = \tan x$.

However, $\tan\frac{x}{2} = \frac{\sin x}{1+\cos x}$.

thus
$$\frac{\sin(2x)}{1+\cos(2x)} = \tan\left(\frac{2x}{2}\right)$$
$$= \tan(x).$$

Example without using a calculator,

$\cos\left(2\cos^{-1}\left(\frac{3}{5}\right)\right)$ Put $\theta = \cos^{-1}\left(\frac{3}{5}\right)$.

$$= \cos(2\theta) = 2\cos^2\theta - 1.$$
$$= 2\cos\left(\cos^{-1}\left(\frac{3}{5}\right)\right)^2 - 1$$
$$= 2\times\left(\frac{3}{5}\right)^2 - 1 = 2\cdot\frac{9}{25} - 1$$
$$= \frac{18-25}{25} = -\frac{7}{25}$$

6.4 Product-Sum and Sum-Product Identities

1. $\sin x \cos y = \frac{1}{2}\left[\sin(x+y) + \sin(x-y)\right]$

Prof

$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$
$$\sin(x-y) = \sin x \cos y - \cos x \sin y$$

Add

$$\sin(x+y) + \sin(x-y) = 2\sin x \cos y$$

Thus $\sin x \cos y = \frac{1}{2}\left(\sin(x+y) + \sin(x-y)\right)$

2. $\cos x \sin y = \frac{1}{2}\left[\sin(x+y) - \sin(x-y)\right]$
3. $\sin x \sin y = \frac{1}{2}\left[\cos(x-y) - \cos(x+y)\right]$
4. $\cos x \cos y = \frac{1}{2}\left[\cos(x+y) + \cos(x-y)\right]$

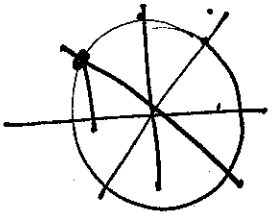
Example1 Write product as sum or difference

$$\sin 3m \cos m = \frac{1}{2}\left(\sin(3m+m) + \sin(3m-m)\right)$$
$$= \frac{1}{2}\sin(4m) + \sin(2m).$$

Example2. Find

$$\sin\frac{13\pi}{24}\ \text{co}\,\frac{5\pi}{24} = \frac{1}{2}\left(\sin\left(\frac{13\pi}{24}+\frac{5\pi}{24}\right)+\sin\left(\frac{13\pi}{24}-\frac{5\pi}{24}\right)\right)$$

$$= \frac{1}{2}\left(\sin\frac{18\pi}{24}+\sin\frac{8\pi}{24}\right)$$



$$= \frac{1}{2}\left(\sin\frac{3\pi}{4}+\sin\frac{\pi}{3}\right)$$

$$= \frac{1}{2}\left(+\frac{\sqrt{2}}{2}+\frac{\sqrt{3}}{2}\right)$$

$$= \frac{1}{2}\left(\frac{\sqrt{3}+\sqrt{2}}{2}\right)$$

$$= \frac{\sqrt{3}+\sqrt{2}}{4} = \frac{\sqrt{3}+\sqrt{2}}{4}$$

(*) $\sin x \cos y = \frac{1}{2}\left[\sin(x+y) + \sin(x-y)\right]$

Put $\theta = x+y$
$\alpha = x-y$ $\Rightarrow \begin{cases} x+y=\theta \\ x-y=\alpha \end{cases}$

$$\overline{2x = \theta+\alpha} \Rightarrow x = \frac{\theta+\alpha}{2},\ y = \frac{\theta-\alpha}{2}$$

then,

(*) becomes

$$\sin\left(\frac{\theta+\alpha}{2}\right)\cos\left(\frac{\theta-\alpha}{2}\right) = \frac{1}{2}\left(\sin\theta + \sin\alpha\right)$$

thus, $\sin\theta + \sin\alpha = 2\sin\left(\frac{\theta+\alpha}{2}\right)\cos\left(\frac{\theta-\alpha}{2}\right)$

Sum- Product Identity

$$\sin x + \sin y = 2 \sin \frac{x+y}{2} \cos \frac{x-y}{2}$$

$$\sin x - \sin y = 2 \cos \frac{x+y}{2} \sin \frac{x-y}{2}$$

$$\cos x + \cos y = 2 \cos \frac{x+y}{2} \cos \frac{x-y}{2}$$

$$\cos x - \cos y = -2 \sin \frac{x+y}{2} \sin \frac{x-y}{2}$$

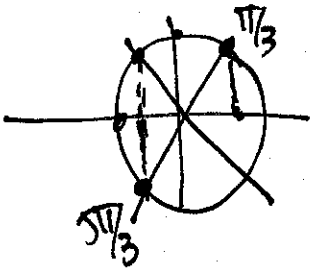
Ex 3 Compute $\cos 105^\circ + \cos 15^\circ$

$$2\cos\left(\frac{120}{2}\right)\cos\left(\frac{90}{2}\right) = 2\cos(60)\cos(45)$$

$$= 2\,\frac{1}{2}\cdot\frac{\sqrt{2}}{2}\ .$$

$$= \frac{\sqrt{2}}{2}$$

Find $\cos(19\pi/12) + \cos(\pi/12)$



$$= 2\cos\left(\frac{20\pi}{12}\right)\cos\left(\frac{18\pi}{12}\right)$$

$$= 2\cos\left(\frac{5\pi}{6}\right)\cos\left(\frac{3\pi}{4}\right)$$

$$= \cancel{2}\left(-\frac{\sqrt{3}}{\cancel{2}}\right)\left(-\frac{\sqrt{2}}{2}\right)$$

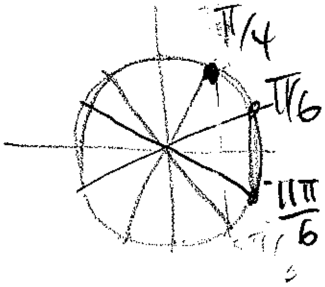
$$= \frac{\sqrt{6}}{2}\ .$$

6.5 Trigonometric Equations

Questions How do we solve the equation

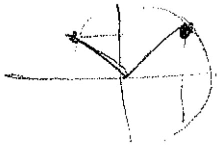
$$\sin(x) = \frac{1}{2} \Rightarrow x = \frac{\pi}{6} + 2k\pi,\ k \in \mathbb{Z}$$

$$x = \frac{11\pi}{6} + 2k\pi;\ k \in \mathbb{Z}$$



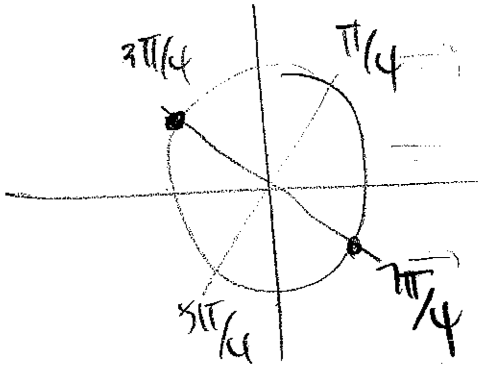
Example $5 + 5\tan\theta = 0$, θ is in degree

$\tan\theta = -1$



$$\frac{\sin\theta}{\cos\theta} = -1 \Rightarrow \sin\theta = -\cos\theta$$

$$\theta = \frac{3\pi}{4} + k\pi \quad k \in \mathbb{Z}$$



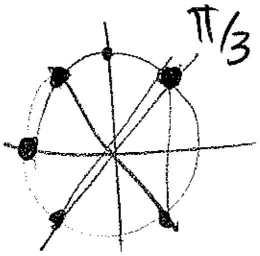
Example Solve $2\cos^2 x - \cos x = 0$

Put $u = \cos x \Rightarrow 2u^2 - u = 0$

$\Rightarrow u(2u-1) = 0$

$\Rightarrow u = 0$ or $2u - 1 = 0$

$\Rightarrow u = 0$ or $u = \frac{1}{2}$

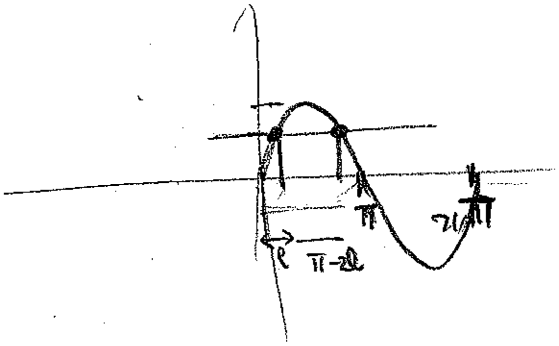


$\Rightarrow \cos x = 0$ or $\cos x = \frac{1}{2}$

$\Rightarrow x = \frac{\pi}{2} + k\pi$ or $x = \frac{\pi}{3} + 2k\pi$ or $x = \frac{5\pi}{3} + 2k\pi$

Example Solve $3\cos^2 x + 8\sin x = 7$

$3(1 - \sin^2 x) + 8\sin x = 7$



$3 - 3\sin^2 x + 8\sin x = 7$

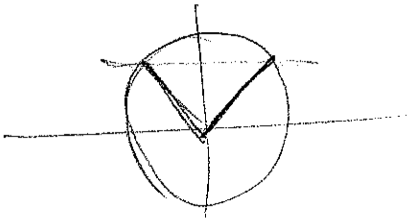
$-3\sin^2 x + 8\sin x - 4 = 0$

$-3u^2 + 8u - 4 = 0$

$u = \frac{2}{3}$ or $u = 2 \Rightarrow \sin x = \frac{2}{3}$

$\Rightarrow x = \arcsin\left(\frac{2}{3}\right)$ or

$x = \pi - \arcsin\left(\frac{2}{3}\right)$



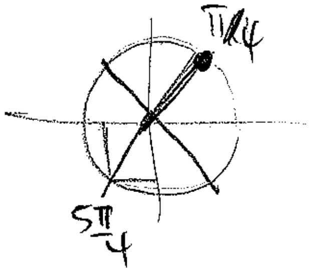
Example Solve $\sin^2 x = \frac{1}{2}\sin 2x, \quad 0 \le x \le 2\pi$

$$\sin^2 x = \frac{1}{2}\, 2\sin x \cos x$$

$$\sin^2 x - \sin x \cos x = 0$$

$$\sin x(\sin x - \cos x) = 0$$

$$\sin x = 0 \text{ or } \sin x - \cos x = 0$$



For $\sin x = 0$, $x = 0, \pi$

For $\sin x - \cos x = 0$, $\sin x = \cos x \Rightarrow x = \frac{\pi}{4}$ or $x = \frac{5\pi}{4}$

The solution set $\{0, \pi, \frac{\pi}{4}, \frac{5\pi}{4}\}$

Example $\sin x + \cos x = \sqrt{2}$

$$\sin x = \sqrt{2} - \cos x$$

$$\sin^2 x = 2 - 2\sqrt{2}\cos x + \cos^2 x$$

$$(1 - \cos^2 x) = 2 - 2\sqrt{2}\cos x + \cos^2 x$$

$$2 - 2\sqrt{2}\cos x + \cos^2 x + \cos^2 x - 1 = 0$$

$$2\cos^2 x - 2\sqrt{2}\cos x + 1 = 0$$

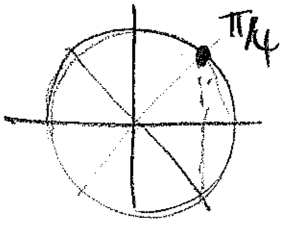
$$2U^2 - 2\sqrt{2}\,U + 1 = 0 \quad , \text{ For } U = \cos x$$

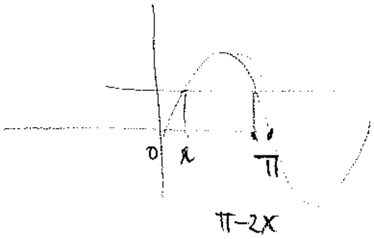
$$U^2 - \sqrt{2}\,U + \left(\frac{1}{\sqrt{2}}\right)^2 = 0 \Rightarrow \left(U - \frac{1}{\sqrt{2}}\right)^2 = 0 \Rightarrow U = \frac{1}{\sqrt{2}}$$

$\cos x = \frac{\sqrt{2}}{2}$

$x = \frac{\pi}{4} + 2k\pi$ or

$x = -\frac{\pi}{4} + 2k\pi, \quad k \in \mathbb{Z}$





$2x + y = \pi$

Example $\sin\frac{\theta}{2} - \cos\theta = 0$

One way

$\sin\frac{\theta}{2} = \sin\left(\frac{\pi}{2} - \theta\right)$

$\frac{\theta}{2} = \frac{\pi}{2} - \theta + 2k\pi \Rightarrow \theta + \frac{\theta}{2} = \frac{\pi}{2} + 2k\pi, \quad k \in \mathbb{Z}$

$\Rightarrow \frac{3\theta}{2} = \frac{\pi}{2} + 2k\pi$

$\Rightarrow \theta = \frac{\pi}{3} + \frac{4k\pi}{3}$

$\sin\frac{\theta}{2} = \cos\theta$

$\sin\frac{\theta}{2} = \sin\left(\frac{\pi}{2} - \theta\right)$

$\frac{\theta}{2} = \frac{\pi}{2} - \theta + 2k\pi$

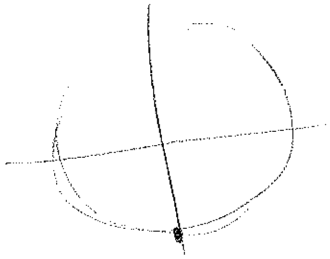
Example Solve $\sec x + \tan x = 1, \quad 0 \le x \le 2\pi$

$\frac{1}{\cos x} + \frac{\sin x}{\cos x} = 1$

$\sin x + 1 = \cos x$

$\cos^2 x = 1 + 2\sin x + \sin^2 x$

$1 - \sin^2 x = 1 + 2\sin x + \sin^2 x \quad ($



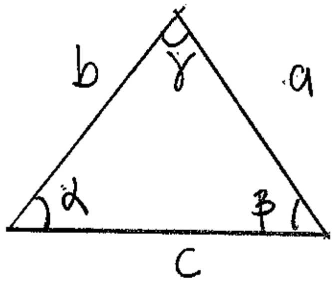
$2\sin^2 x + 2\sin x = 0$

$\sin x(\sin x + 1) = 0$

$x = 0, \pi, 2\pi$ or $\sin x = -1, x = \frac{3\pi}{2}$

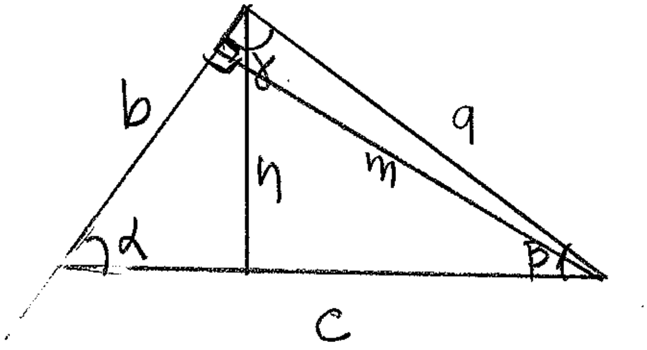
7.1

Law of Sines



$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$$

Proof



$$\sin\alpha = \frac{h}{b} \Rightarrow b\sin\alpha = h$$

$$\sin\beta = \frac{h}{a} \rightarrow a\sin\beta = h$$

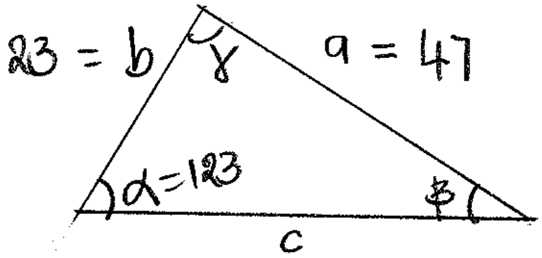
Thus, $b\sin\alpha = a\sin\beta \Rightarrow \frac{\sin\alpha}{a} = \frac{\sin\beta}{b}$ ①

Now, $\sin\gamma = \frac{m}{a}$ and $\sin\alpha = \frac{m}{c}$

$$m = a\sin\gamma = c\sin\alpha \Rightarrow \frac{\sin\gamma}{c} = \frac{\sin\alpha}{a}$$

Using ① we obtain $\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$

Example Solve the triangle



$$\frac{\sin 123}{47} = \frac{\sin\beta}{23} = \frac{\sin\gamma}{c}$$

$$\sin\beta = \frac{23 \times \sin 123^\circ}{47}$$

$180 \to \pi$
$123 \to ?$

$$= \frac{23 \times \sin\frac{123\pi}{180}}{47} = 0.410$$

$$\beta = \arcsin\left(\frac{23\sin\left(\frac{123\pi}{180}\right)}{47}\right)$$

$$= 0.423 \text{ rad}$$

$180 \to \pi$
$? \to 0.423$

$$= \frac{0.423 \times 180}{\pi} \text{ rad}$$

$$\approx 24^\circ$$

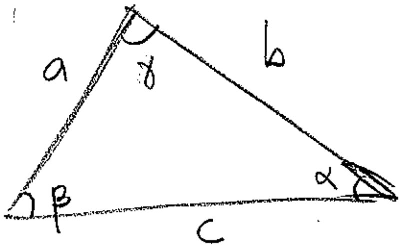
$$\gamma = 180 - 123 - 24 = 33 \Rightarrow \frac{\sin 33}{c} = \frac{\sin 123}{47}$$

$$\Rightarrow c = \frac{47\sin 33}{\sin 123}$$

$$c = \frac{47 \sin 33}{\sin 123} = 30.522$$

<u>Example</u> Using the Law of sine show that

$$\frac{a-b}{a+b} = \frac{\tan \frac{\alpha-\beta}{2}}{\tan \frac{\alpha+\beta}{2}}$$



Recall that if

$$\frac{A}{B} = \frac{C}{D} \rightarrow \frac{A-C}{A+C} = \frac{B-D}{B+D}$$

Since

$$\frac{A-C}{A+C} = \frac{\left(\frac{BC}{D} - C\right)}{\left(\frac{BC}{D} + C\right)} = \frac{\left(\frac{B}{D} - 1\right)C}{\left(\frac{B}{D} + 1\right)C}$$

$$= \frac{\frac{B-D}{D}}{\frac{B+D}{D}} = \frac{B-D}{B+D}.$$

Next,

$$\frac{\tan \frac{\alpha-\beta}{2}}{\tan \frac{\alpha+\beta}{2}} = \frac{\frac{\sin \frac{\alpha-\beta}{2}}{\cos \frac{\alpha-\beta}{2}}}{\frac{\sin \frac{\alpha+\beta}{2}}{\cos \frac{\alpha+\beta}{2}}} = \frac{\sin\left(\frac{\alpha-\beta}{2}\right) \cos\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right) \sin\left(\frac{\alpha+\beta}{2}\right)}$$

$$= \frac{\frac{1}{2}\sin(\alpha) + \sin(-\beta)}{\frac{1}{2}\sin\alpha - \sin(-\beta)}$$

$$\frac{1}{2} = \frac{2}{4}$$

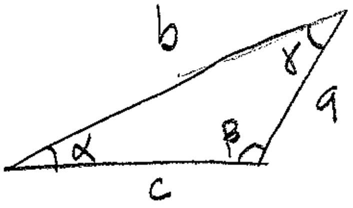
$$-\frac{2}{6} = -\frac{1}{3}$$

$$= \frac{\sin\alpha - \sin\beta}{\sin\alpha + \sin\beta} = \frac{a-b}{a+b}$$

Since $\frac{\sin\alpha}{a} = \frac{\sin\beta}{b}$.

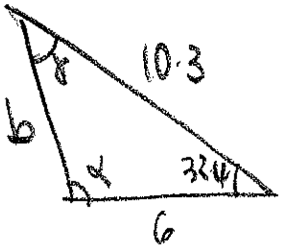
$$\frac{A}{B} = \frac{C}{D} \Rightarrow \frac{B-D}{B+D} = \frac{A-C}{A+C}$$

7.2 Law of Cosines

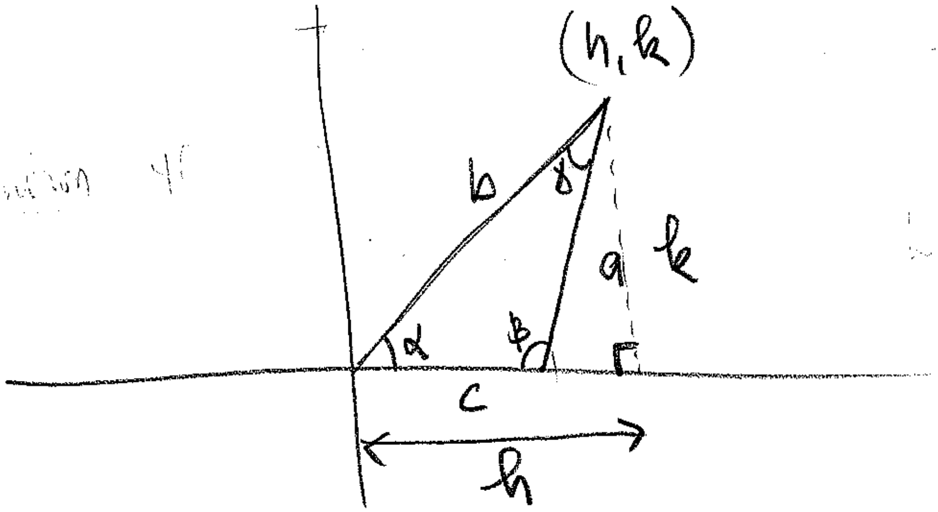


$$a^2 = b^2+c^2 - 2bc\cos\alpha$$
$$b^2 = a^2+c^2 - 2ac\cos\beta$$
$$c^2 = a^2+b^2 - 2ab\cos\gamma$$

Example Solve the following triangle



$$b^2 = 6^2 + 10.3^2 - 2(6)(10.3)\cos(32.4°)$$
$$= 5.96 \text{ cm}$$



we know that $a^2 = k^2 + (h-c)^2$

$$= h^2 - 2hc + c^2 + k^2$$

but $b^2 = h^2 + k^2$.

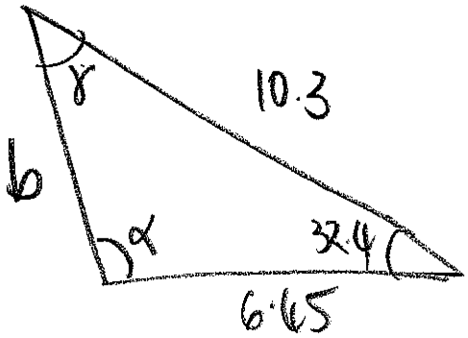
Thus $a^2 = h^2 + k^2 - 2hc + c^2$

$$= b^2 + c^2 - 2hc$$. (Get rid of h)

However $\cos\alpha = \frac{h}{b} \Rightarrow h = b\cos\alpha$.

It follows that $a^2 = b^2 + c^2 - 2bc\cos\alpha$.

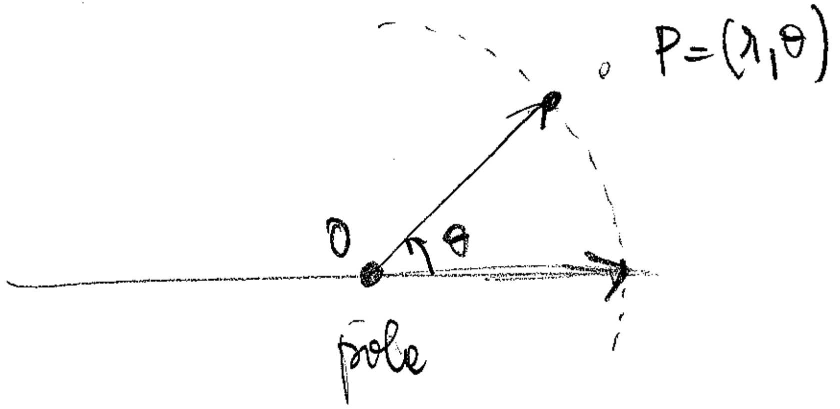
Example Solve the triangle



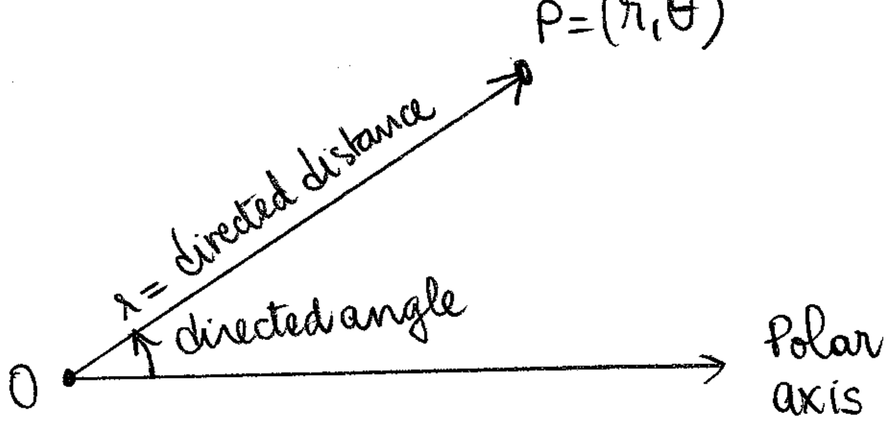
$$b^2 = (6.45)^2 + (10.3)^2 - 2(6.45)(10.3)\cos(32.4)$$

7.4 Polar coordinates and graphs

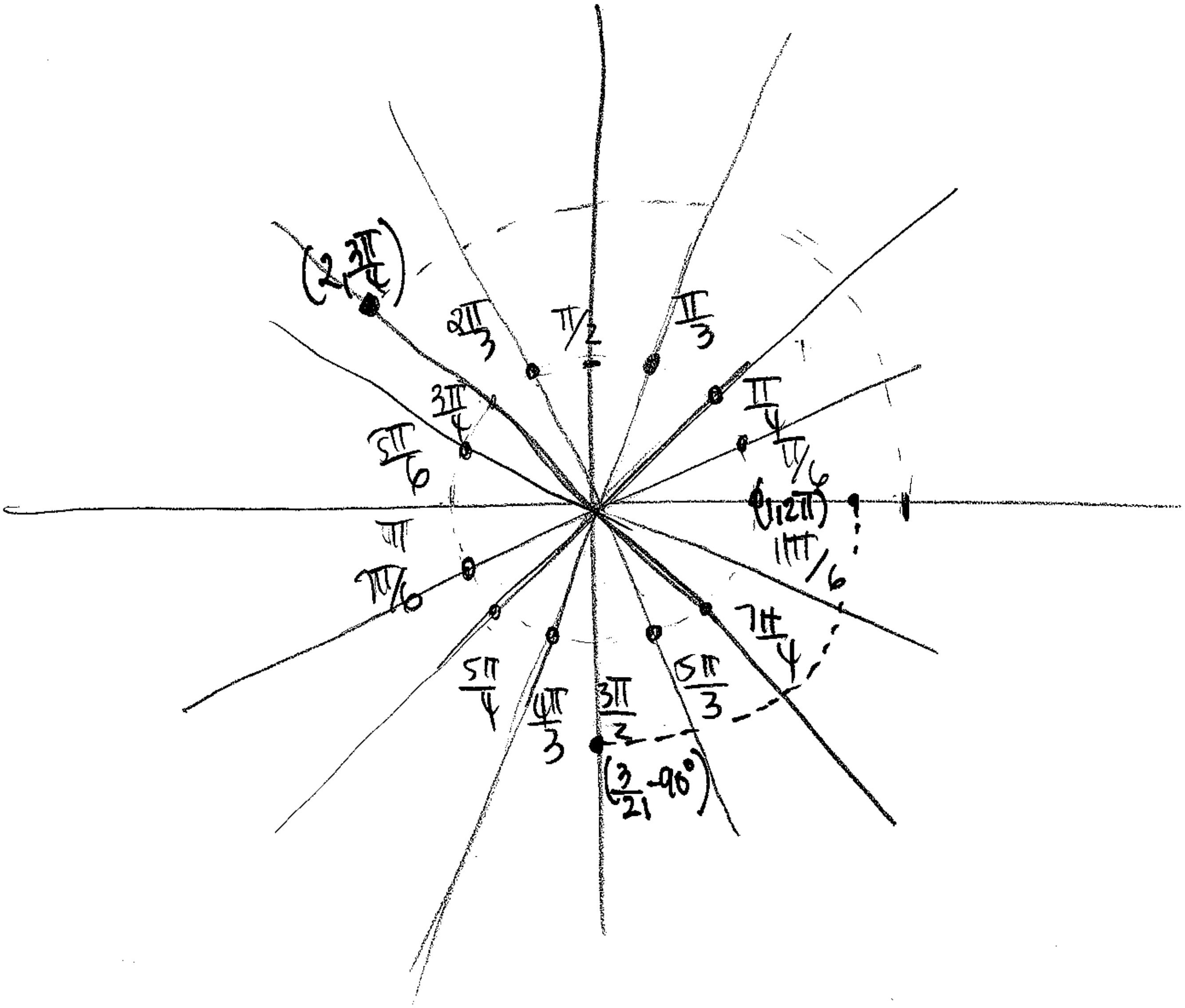
We consider a fixed point on a plane called "pole". From the fixed point, we draw a ray called "polar axis." Any point in a plane can be obtained my moving along the polar axis and by rotating the axis by a certain angle θ.



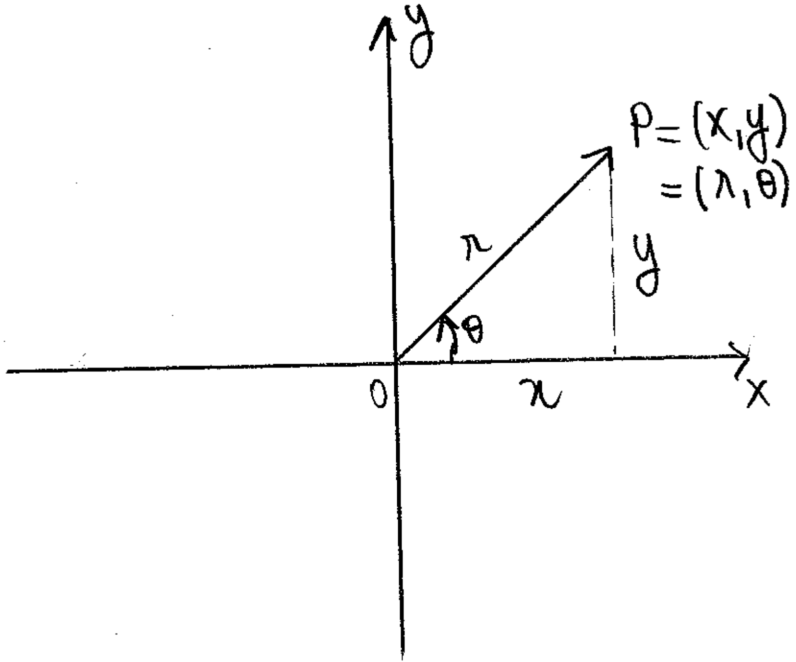
(r,θ) are the polar coordinates of P, where r is the is the directed distance from P to O and θ is the directed angle of rotation



Example Plot the points $(1, 2\pi)$, $(2, \frac{3\pi}{4})$, $(\frac{3}{2}, -90°)$



Polar - Rectangular Relationships (coordinates conversion)



we always assume $\theta \in (-\pi, \pi]$

We have $\frac{x}{r} = \cos\theta \Rightarrow x = r\cos\theta$ and $\frac{y}{r} = \sin\theta \Rightarrow y = r\sin\theta$

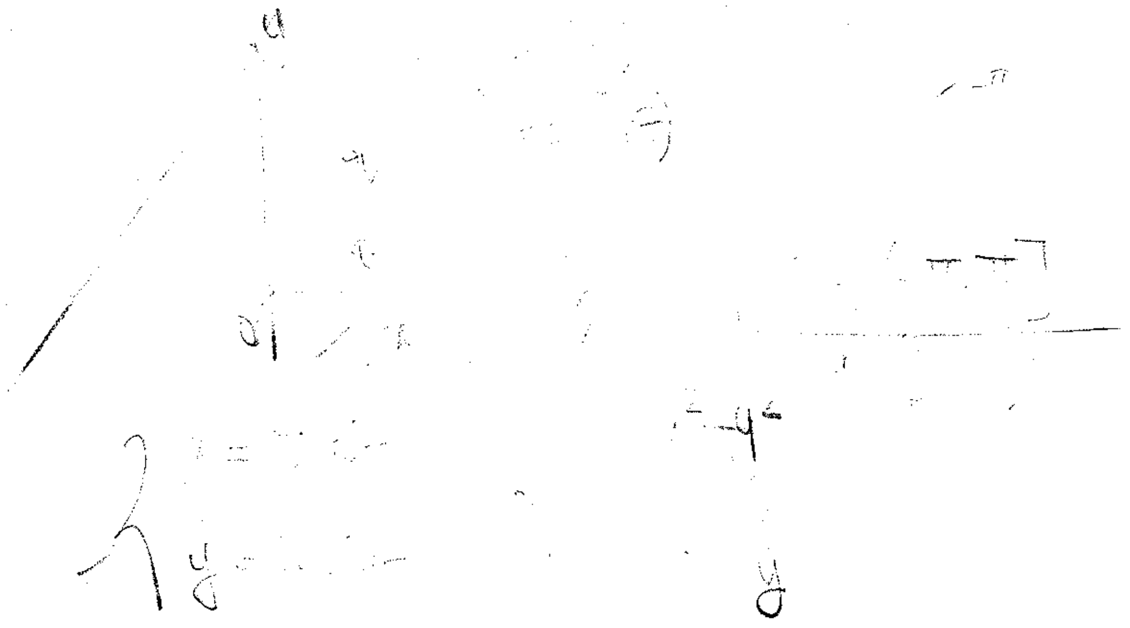
$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$ (Conversion Polar to rectangle coordinates)

$$\begin{cases} r^2 = x^2 + y^2 \\ \tan\theta = \frac{y}{x} \end{cases}$$ (Rectangular to polar coordinate conversion)

the sign of r and the choice of θ place the point (r, θ) in the same quadrant as (x, y)

$$\boxed{\begin{cases} r = \pm\sqrt{x^2+y^2} \\ \theta = \arctan\frac{y}{x} \text{ or } \arctan\frac{y}{x} + \pi \end{cases}}$$

[illegible]



Example Convert $(-4, 2)$ to rectangular coordinates

$r = -4, \theta = 2$

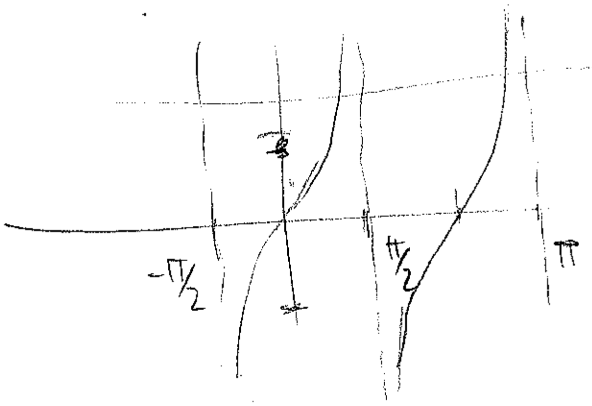
$x = -4 \cos 2$

$y = -4 \sin 2$

$(-4\cos 2, -4 \sin 2) = (1.66, -3.63)$

Convert $(-3, -5)$ to polar coordinates

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \Rightarrow \begin{cases} -3 = r\cos\theta \\ -5 = r\sin\theta \end{cases} \Rightarrow \tan\theta = \frac{5}{3}$$



Let $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ such that $\tan\alpha = \frac{5}{3}$

$$\tan\alpha = \frac{5}{3} \Rightarrow \alpha = 1.03 \text{ rad} = 59.03^\circ \quad , \alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$180 \to \pi$

$x \to 1.03$

However θ belongs to the 3rd quadrand $\Rightarrow$

$$\theta = \pi + 1.03 = 180 + 59.03^\circ = 239.03^\circ$$

Since $x = r\cos\theta \Rightarrow r = \frac{x}{\cos\theta} = \frac{-3}{\cos(\pi + 1.03)}$

$$= 5.82$$

$(-3, -5)$ in polar coordinate is

$$\left(5.82,\ 239.03^\circ\right)$$

Example Convert to polar form

$x^2 + y^2 - 3y = 0$

$(x-0)^2 + (y - \frac{3}{2})^2 = \frac{9}{4}$

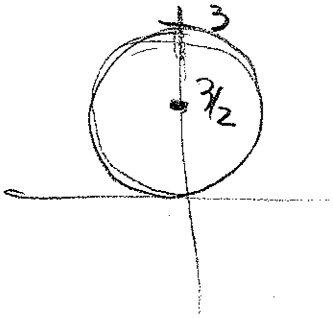
$r^2 = x^2 + y^2 \Rightarrow x^2 + y^2 - 3y = r^2 - 3(r\sin\theta) = 0$

$\Rightarrow r^2 - 3r\sin\theta = 0$

$\Rightarrow r(r - 3\sin\theta) = 0$

$\Rightarrow r = 0$ or $r - 3\sin\theta = 0$

$\Rightarrow r = 0 \ , \ r = 3\sin\theta$



Example. Convert to rectangular form and plot

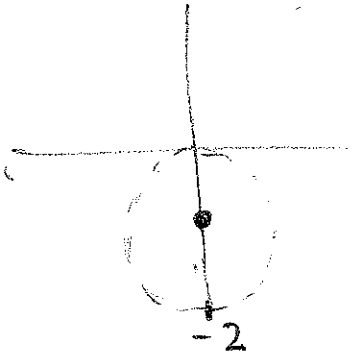
$r + 2\sin\theta = 0$

$r = -2\sin\theta \Rightarrow r^2 = -2r\sin\theta$

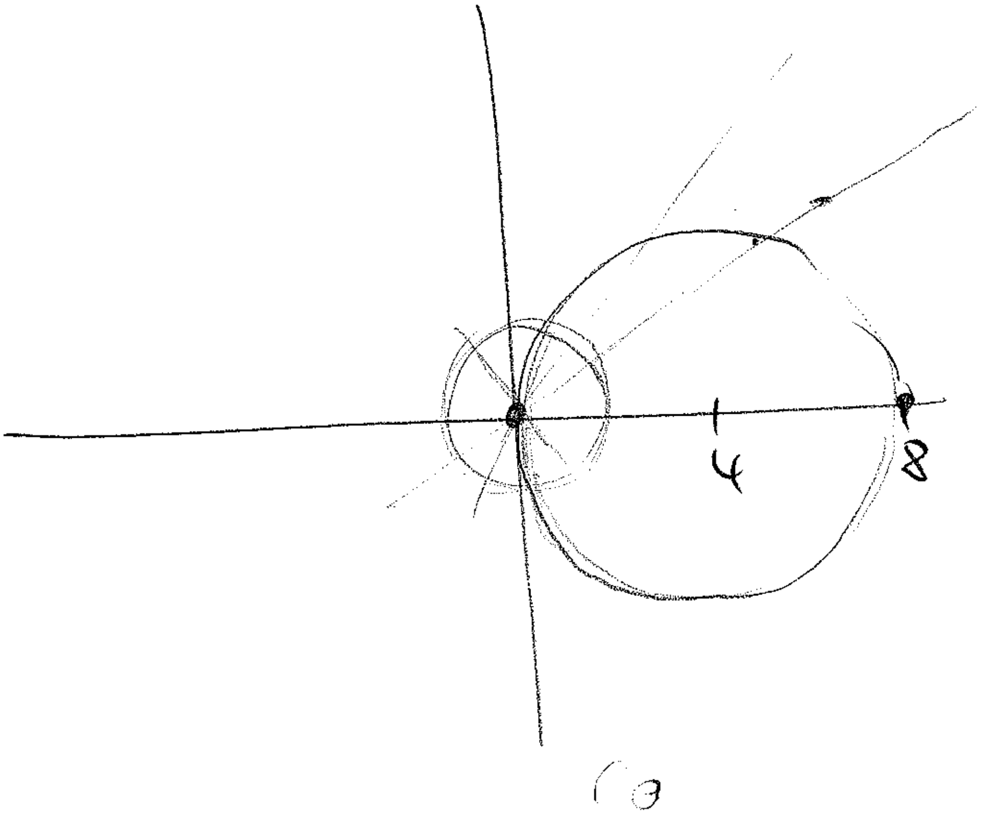
$\Rightarrow x^2 + y^2 = -2y$

$\Rightarrow x^2 + y^2 + 2y = 0$

$(x-0)^2 + (y+1)^2 = 1$



Example graph $r = 8\cos\theta$



θ	r
0	8
$\frac{\pi}{6}$	6.9
$\frac{\pi}{3}$	4.0
$\pi/2$	0
$\frac{2\pi}{3}$	-4
$\frac{5\pi}{6}$	-6.9
π	-8

Rapid polarsketching

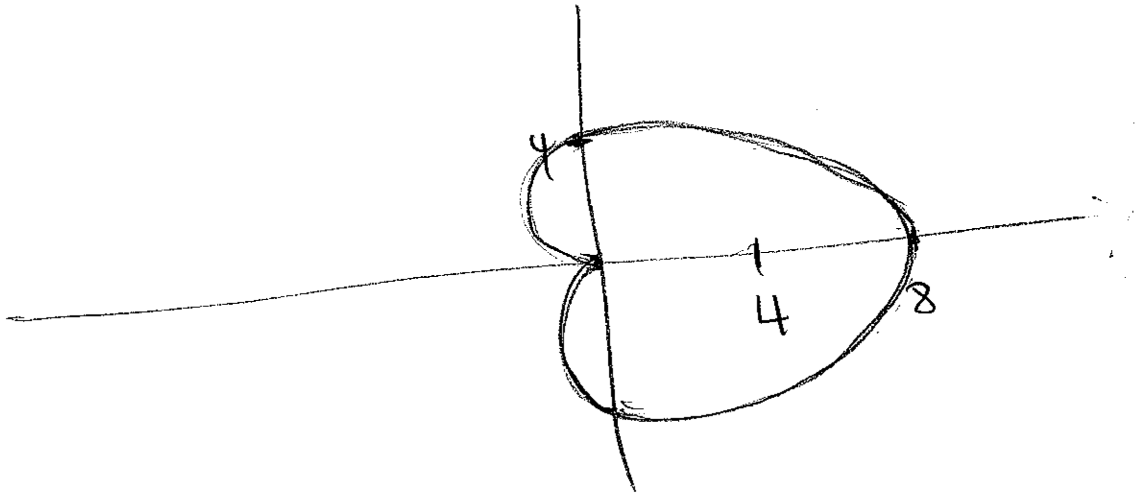
$r = 4 + 4\cos\theta$

From 0 to $\frac{\pi}{2}$, r varies from 8 to 4

From $\frac{\pi}{2}$ to π, r varies from 4 to 0

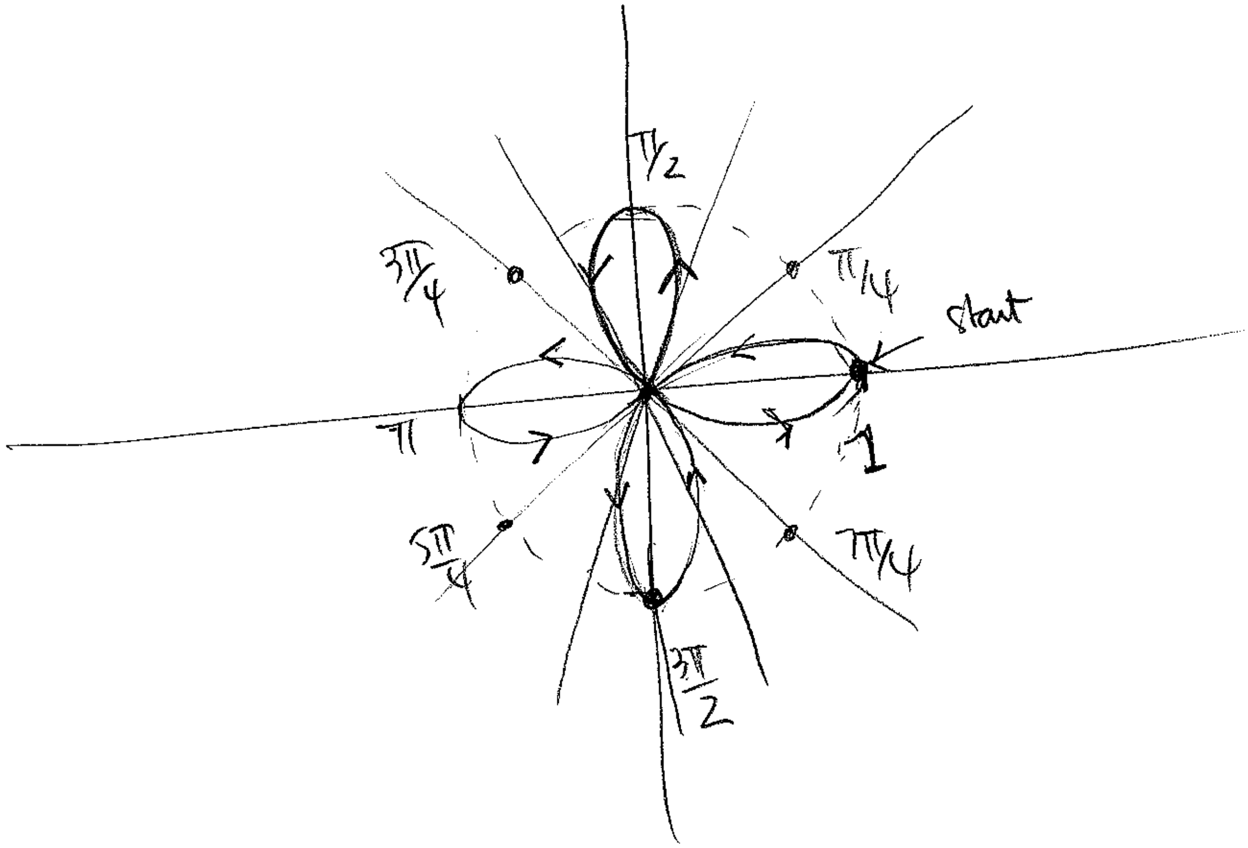
From π to $\frac{3\pi}{2}$, r varies from 0 to 4

From $\frac{3\pi}{2}$ to 2π, r varies from 4 8



Cardioid

Example sketch $r = \cos 2\theta$



Four-leafed rose

θ	r
$0 - \frac{\pi}{4}$	$1 - 0$
$\frac{\pi}{4} - \frac{\pi}{2}$	$0 - -1$
$\frac{\pi}{2} - \frac{3\pi}{4}$	$-1 - 0$
$\frac{3\pi}{4} - \pi$	$0 - 1$
$\pi - \frac{5\pi}{4}$	$1 - 0$
$\frac{5\pi}{4} - \frac{3\pi}{2}$	$0 - (-1)$
$\frac{3\pi}{2} - \frac{7\pi}{4}$	$-1 - 0$
$\frac{7\pi}{4} - 2\pi$	$0 - 1$

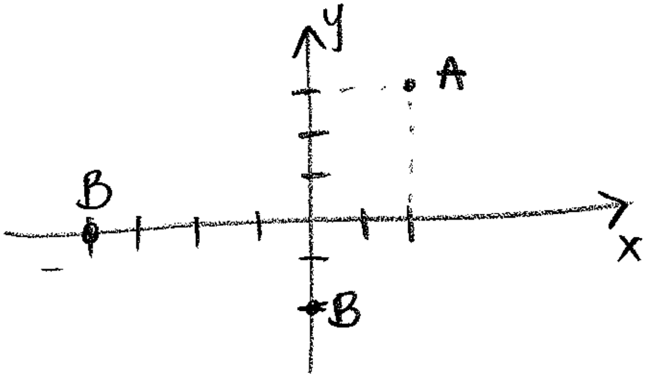
7.5 - Complex Numbers and De Moivre's theorem

Recall that any complex number can be written as
$z = x + iy$.

We associate $z = x+iy$ to the ordered pair (x,y). With such identification, the cartesian plane becomes a complex plane. the x-axis is called the real axis, and the y-axis is called the imaginary axis.

Example Plot the complex numbers in the plane

$A = 2+3i$, $B = -4$, $C = -2i$



Polar form

Let z be a complex number, $z = x+iy$

Let $r = \sqrt{x^2+y^2}$ and $x = r\cos\theta$, $y = r\sin\theta$, $\tan\theta = \frac{y}{x}$, $x \neq 0$

then $z = x + iy = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta)$

$= re^{i\theta}$, where $e^{i\theta} = \cos\theta + i\sin\theta$.

r is called the modulus or absolute value of z

θ is called the argument of z.

We say $r = \mathrm{mod}\, z)$, $\theta = \arg z$.

<u>Remark</u> (r,θ) and $(r,\theta+2\pi)$ represent the same point in the complex plane.

Notice $z = re^{i\theta}$, $re^{i(\theta+2\pi)} = re^{i\theta}e^{i2\pi}$

but $e^{i(2\pi)} = \cos(2\pi) + i\sin 2\pi$

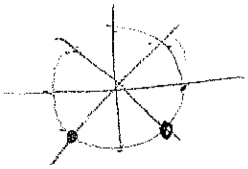
$= 1$.

Thus, $z = re^{i(\theta+2\pi)} = re^{i\theta}\cdot 1 = re^{i\theta}$.

So, $re^{i\theta} = re^{i(\theta+2\pi)}$

We will choose θ so that $\theta \in (-\pi, \pi]$ ← Important

<u>Example</u> Write in polar forms



$z_1 = 1 - i$, $x = 1$, $y = -1$

$r = \sqrt{1+1} = \sqrt{2}$. $\tan\theta = -1$

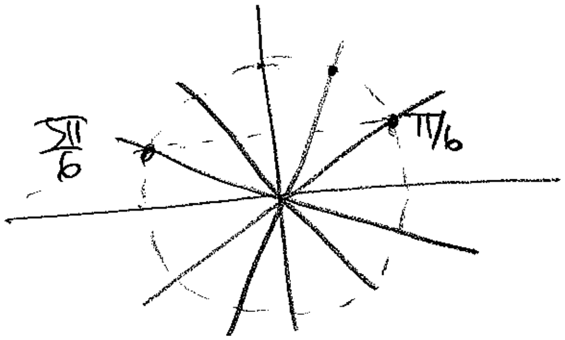
θ is in the 4th quadrand thus $\theta = \frac{7\pi}{4} = -\frac{\pi}{4}$

$z_1 = \sqrt{2}e^{-i\frac{\pi}{4}}$

$z_2 = -\sqrt{3} + i$

$x = -\sqrt{3}$, $y = 1$, $r = \sqrt{3+1} = 2$

$\cos\theta = -\frac{\sqrt{3}}{2}$ and $\sin\theta = \frac{1}{2}$ $\Rightarrow$ $\theta = \frac{5\pi}{6}$



$z_2 = 2e^{i\frac{5\pi}{6}}$.

Example write in rectangular (complex) form

$z_1 = e^{i\frac{\pi}{2}}$, $r = 1$, $\theta = \frac{\pi}{2}$ $\Rightarrow z_1 = i$

$$z_2 = 2e^{\frac{i\pi}{4}} = 2\left(\frac{\sqrt{2}}{2} + \frac{i\sqrt{2}}{2}\right)$$
$$= \sqrt{2} + i\sqrt{2}$$

$$z_3 = 6e^{\left(\frac{\pi}{6}\right)i} = 6\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right)$$
$$= 6\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right)$$
$$= 3\sqrt{3} + 3i$$

Theorem Product and quotients in polar form

If $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$

1. $z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$
2. $\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$

Proof 1

$$z_1 \cdot z_2 = r_1 e^{i\theta_1} \, r_2 e^{i\theta_2}$$

$$= r_1 (\cos\theta_1 + i\sin\theta_1) \, r_2(\cos\theta_2 + i\sin\theta_2)$$

$$= r_1 r_2 \Big(\cos\theta_1 \cos\theta_2 + i\cos\theta_1 \sin\theta_2 + i\sin\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2$$

$$= r_1 r_2 \Big(\cos\theta_1 \cos\theta_2 + i\left(\cos\theta_1 \sin\theta_2 + \sin\theta_1 \cos\theta_2\right) - \sin\theta_1 \sin\theta_2\Big)$$

$$= r_1 r_2 \Big(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\Big)$$

$$= r_1 r_2 e^{i(\theta_1+\theta_2)}$$

Proof 2

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i\theta_1} e^{-i(\theta_2)} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$$

Example

Let $z_1 = 9e^{165i}$, $z_2 = 3e^{55i}$

$z_1 z_2 = -27e^{220i}$

$\frac{z_1}{z_2} = 3e^{110i}$

Theorem De moivre's theorem

If $z = re^{i\theta}$, then $z^n = r^n e^{in\theta}$.

Example Use Demovre's theorem to compute $(1+i\sqrt{3})^5$

Let $x = 1$, $y = \sqrt{3}$ then $r = \sqrt{3+1} = 2$

$\cos\theta = \frac{1}{2}$, $\sin\theta = \frac{\sqrt{3}}{2} \Rightarrow \theta = \frac{\pi}{3}$.

$1 + i\sqrt{3} = 2e^{i\frac{\pi}{3}}$. Thus $(1+i\sqrt{3})^5 = \left(2e^{i\frac{\pi}{3}}\right)^5$

$= 32e^{\frac{i5\pi}{3}}$

$$(1+i\sqrt{3})^5 = 32\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)$$

$$= 32\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = 16\left(1 - i\sqrt{3}\right)$$

Theorem The nth root theorem.

Let $n>1$, $z = re^{i\theta} \in \mathbb{C}$. Then z has n distinct roots given by

$$z_n = r^{1/n} e^{i(\frac{\theta}{n} + \frac{2k\pi}{n})} , \quad k = 0, 1, \dots, n-1$$

Notice
$$(z_n)^n = \left(r^{\frac{1}{n}} e^{i(\frac{\theta}{n}+\frac{2k\pi}{n})}\right)^n$$
$$= r e^{i(\theta+2k\pi)}$$

Example Find 5 distinct fifth roots of $1+i$.

$x=1$, $y=1$ $\quad r=\sqrt{2}$, $\theta=\frac{\pi}{4}$.

$$1+i = \sqrt{2} e^{i\frac{\pi}{4}} = 2^{\frac{1}{2}} e^{i\frac{\pi}{4}} = 2^{\frac{1}{2}} e^{i(\frac{\pi}{4}+2k\pi)}$$

Assume $z^5 = 2^{\frac{1}{2}} e^{i(\frac{\pi}{4}+2k\pi)}$ $\quad k = 0,1,2,3,4$

$$z = 2^{\frac{1}{10}} e^{i(\frac{\pi}{20}+\frac{2k\pi}{5})} \quad k=0,1,2,3,4$$

$$z_0 = 2^{\frac{1}{10}} e^{i(\frac{\pi}{20})}$$
$$z_1 = 2^{\frac{1}{10}} e^{i(\frac{\pi}{20}+\frac{2\pi}{5})} = 2^{\frac{1}{10}} e^{i(\frac{9\pi}{20})}$$
$$z_2 = 2^{\frac{1}{10}} e^{i(\frac{\pi}{20}+\frac{4\pi}{5})} = 2^{\frac{1}{10}} e^{i(\frac{17\pi}{20})}$$

$$\varepsilon_3 = 2^{\frac{1}{10}} e^{i\left(\frac{\pi}{20}+\frac{6\pi}{5}\right)} = 2^{\frac{1}{10}} e^{i\left(\frac{25\pi}{20}\right)}$$

$$\varepsilon_4 = 2^{\frac{1}{10}} e^{i\left(\frac{\pi}{20}+\frac{8\pi}{5}\right)} = 2^{\frac{1}{10}} e^{i\left(\frac{33\pi}{20}\right)}$$

<u>Example</u> Solve the equation $x^4-1=0$

$$x^4=1 \Rightarrow x^4 = e^{i(2k\pi)}, \quad k \in \{0,1,2,3\}$$

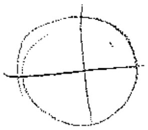
$$\Rightarrow x = e^{i\left(\frac{2k\pi}{4}\right)} \quad k \in \{0,1,2,3\}$$

$$\Rightarrow x = e^{i\left(\frac{k\pi}{2}\right)}$$

$$x = 1, \; e^{i\frac{\pi}{2}}, \; e^{i(\pi)}, \; e^{i\frac{3\pi}{2}}$$

$$x = 1, \; i, \; -1, \; -i$$

the solution set is $\{1, -1, i, -i\}$



<u>Example</u> Solve the equation $x^3+1=0$

$$x^3 = -1 = e^{i(\pi+2k\pi)}, \quad k = 0,1,2$$

$$x = e^{i\left(\frac{\pi}{3}+\frac{2k\pi}{3}\right)}, \quad k = 0,1,2$$

$$x = e^{i\frac{\pi}{3}}, \; e^{i\left(\frac{\pi}{3}+\frac{2\pi}{3}\right)}, \; e^{i\left(\frac{\pi}{3}+\frac{4\pi}{3}\right)}$$

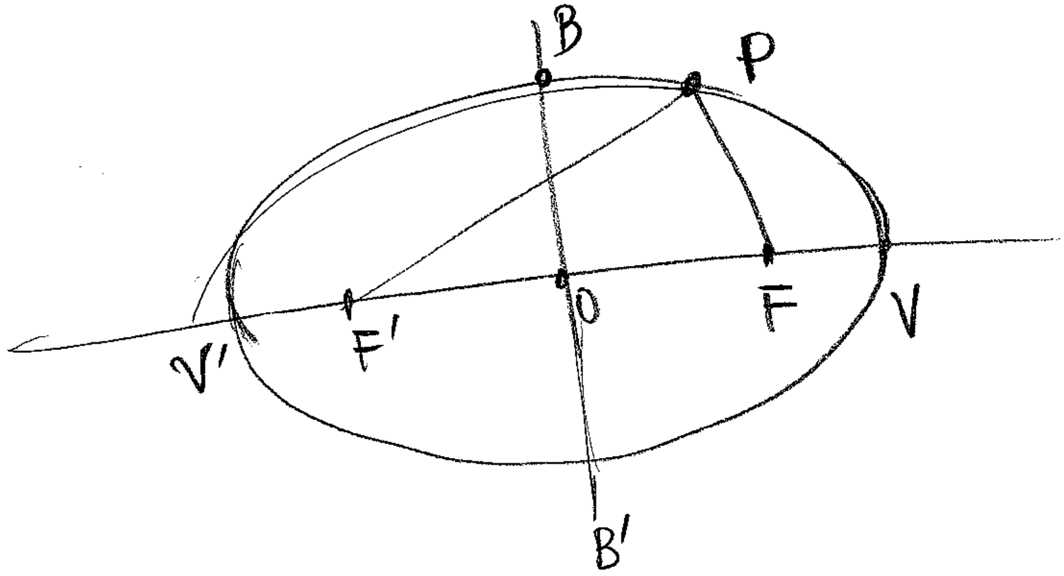
$$= e^{i\frac{\pi}{3}}, \; e^{i\pi}, \; e^{i\frac{5\pi}{3}}$$

The solution set is

$$\left\{ \frac{1}{2} + i\frac{\sqrt{3}}{2}, -1, 1 - i\frac{\sqrt{3}}{2} \right\}$$

11.2 Ellipse

Definition An ellipse is the set of all pts P in a plane such that the sum of the distances from P to two distinct fixed points in the plane is constant



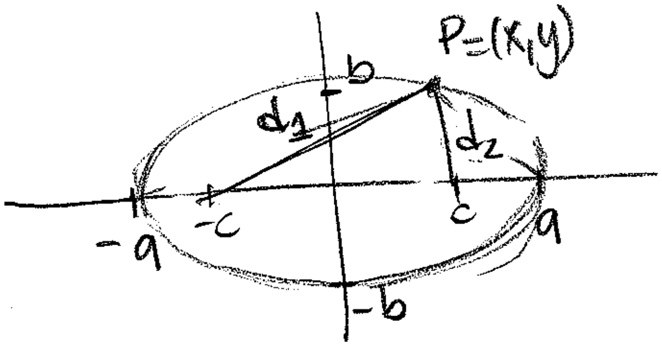
F, F' are called foci.

The segment BB' is called the minor axis

The segment VV' is called the major axis.

The Midpoint of VV' is called the center of the ellipse

Equation of an ellipse



$$d_1 + d_2 = 2a$$

$$d_1 = \sqrt{(x+c)^2 + y^2}$$

$$d_2 = \sqrt{(x-c)^2 + y^2}$$

$$d_1 + d_2 = \sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a$$

$$(x+c)^2 + y^2 + (x-c)^2 + y^2 + 2\sqrt{\left((x+c)^2 + y^2\right)\left((x-c)^2 + y^2\right)} = 2a$$

After simplification, we obtain

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

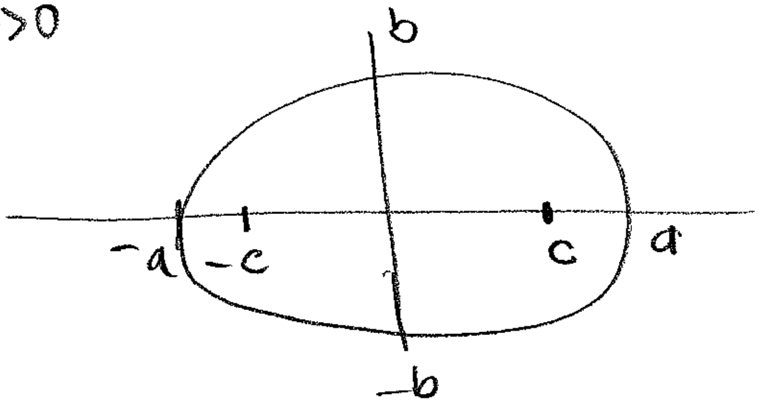
Theorem Standard equations of an Ellipse with center at (0,0)

1. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a > b > 0$

$F' = (-c, 0)$, $F = (c, 0)$

$c^2 = a^2 - b^2$

Major axis is horizontal

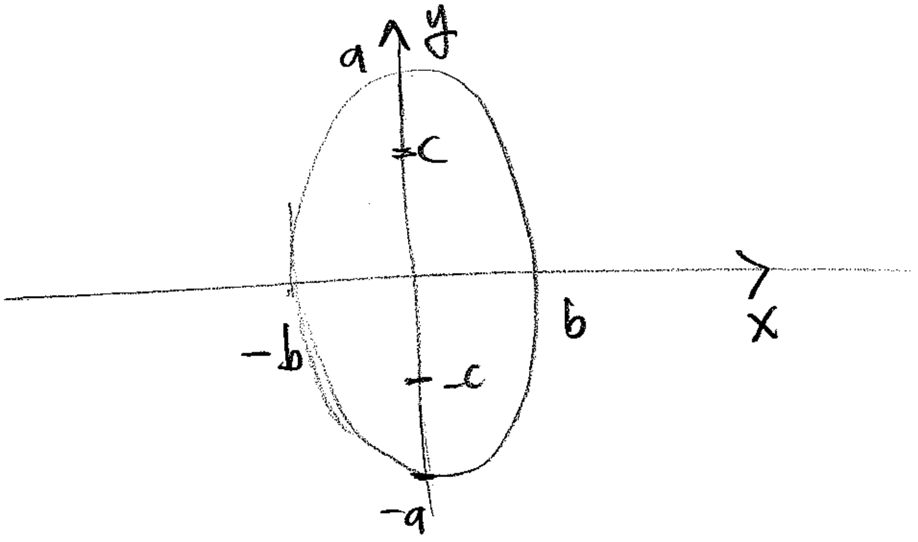


2. $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$, $a > b > 0$

$F' = (0, -c)$, $F = (0, c)$

$c^2 = a^2 - b^2$

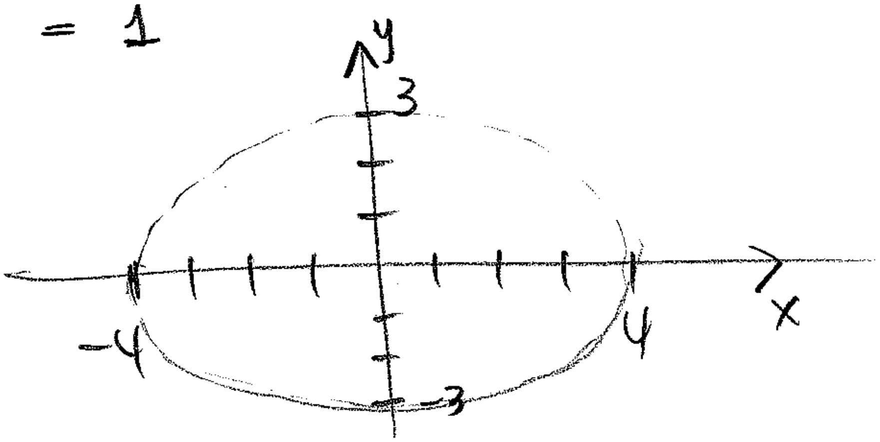
Major axis is vertical



Example Graph the ellipse $9x^2 + 16y^2 = 144$

$$\frac{x^2}{16} + \frac{y^2}{9} = 1$$

$c = \sqrt{16-9} = \sqrt{7}$



Ex Find an equation of an ellipse in the form

$$\frac{x^2}{M}+\frac{y^2}{N}=1 \quad , \; M,N>0$$

a) such that the length major axis $=20$
length minor axis $=12$

$2a=20$ and $2b=12 \Rightarrow a=10, b=6$

$$\frac{x^2}{100}+\frac{y^2}{36}=1$$

b) the length of the major axis $=10$ and
distance from foci $=4$

$c=4$, $2a=10 \Rightarrow a=5$, $c^2=a^2-b^2 \Rightarrow b^2=a^2-c^2$

$\Rightarrow b^2=25-16$

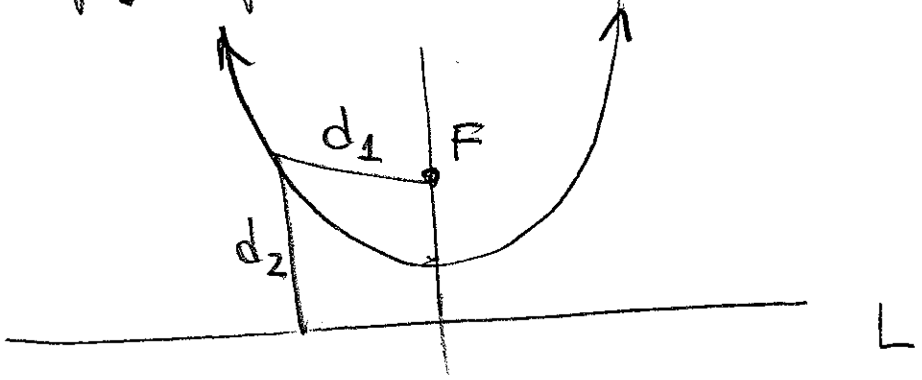
$\Rightarrow b^2=9$

$\Rightarrow b=\pm 3$

$$\frac{x^2}{25}+\frac{y^2}{9}=1$$

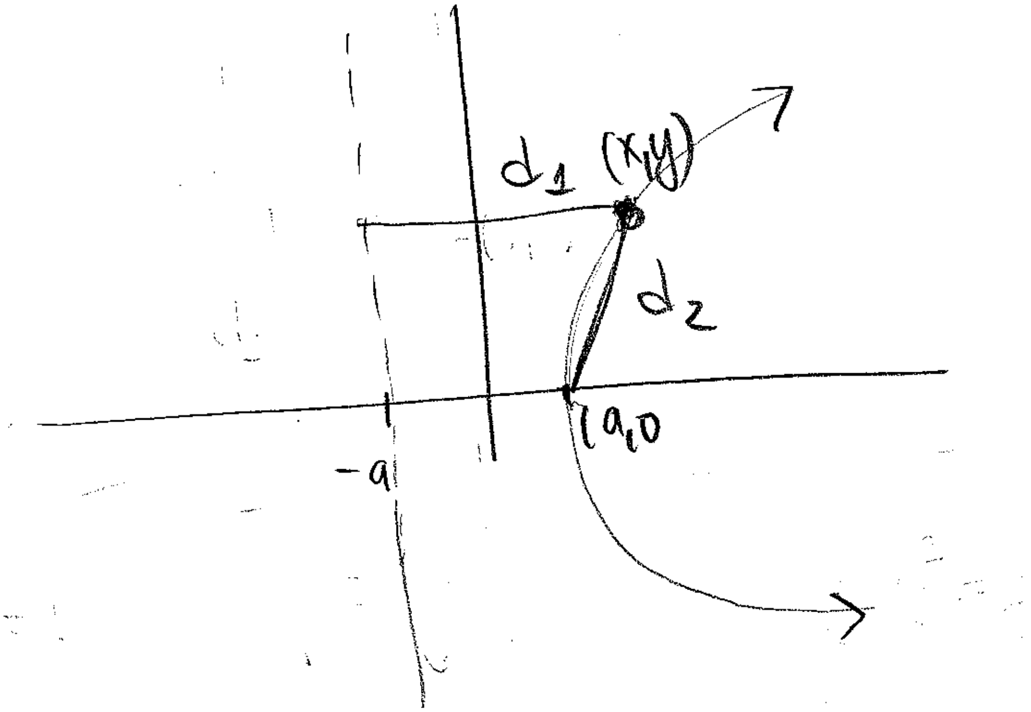
11.1 Conic section ; Parabola

Definition A parabola is the set of all pts equidistant from a fixed point F and a fixed line L



$d_1 = d_2$

F is called the Focus
L is called the directrix.



$$d_1 = \sqrt{(x+a)^2} = d_2 = \sqrt{(x-a)^2+(y)^2}$$

$$(x+a)^2 = (x-a)^2 + y^2$$

$$x^2+2ax+a^2 = x^2-2ax+a^2+y^2$$

$$y^2 = x^2+2ax+a^2-x^2+2ax-a^2$$

$$y^2 = 4ax$$

Theorem

If $y^2 = 4ax$,

Vertex $= (0,0)$

Focus $= (a,0)$

Directrix : $x=-a$

If $x^2=4ay$

Vertex $= (0,0)$

Focus $= (0,a)$

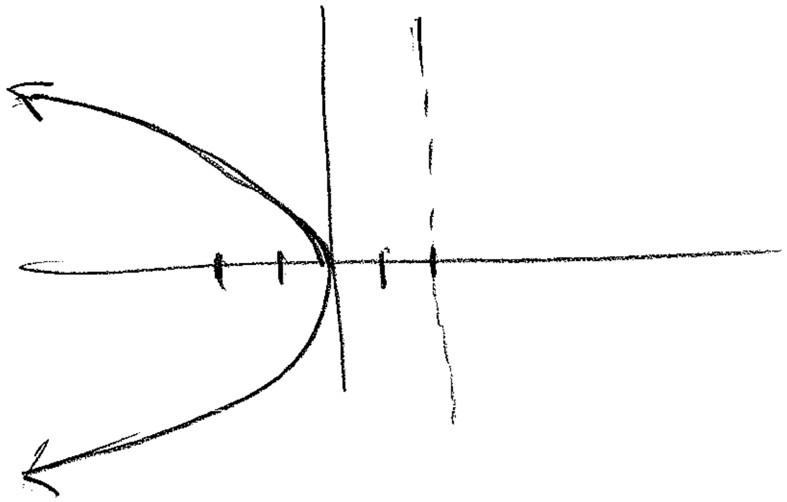
Directrix : $y=-a$

Example Find the focus and directrix for $y^2=-8x$

$$y^2 = 4(-2)x$$

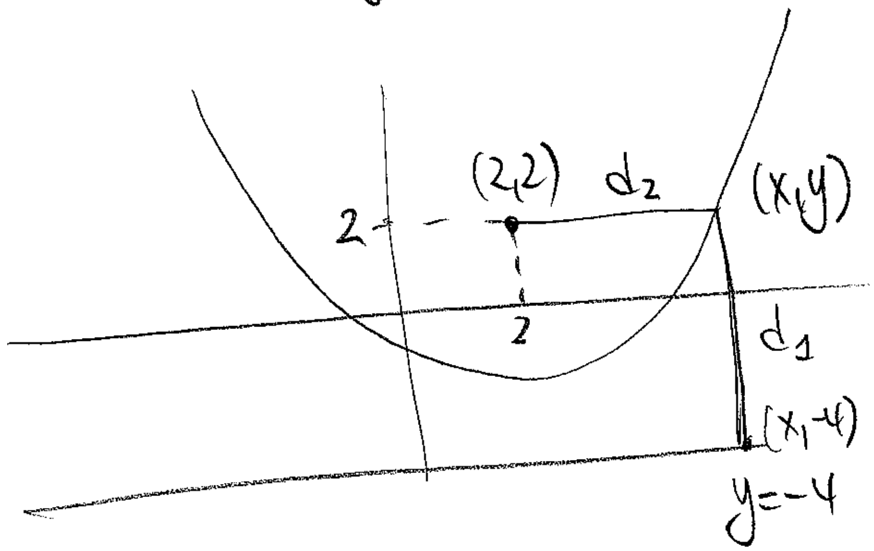
Focus $= (-2,0)$

directrix : $x=2$



__Example__ Use the definition of a parabola and the distance formula to find the equation of a parabola with

Directrix $y=-4$ and focus $(2,2)$



$$d_1 = \sqrt{(y+4)^2}\ ,\ d_2 = \sqrt{(x-2)^2+(y-2)^2}$$

$$(y+4)^2 = (x-2)^2+(y-2)^2$$

$$y^2+8y+16 = x^2-4x+4+y^2-4y+4$$

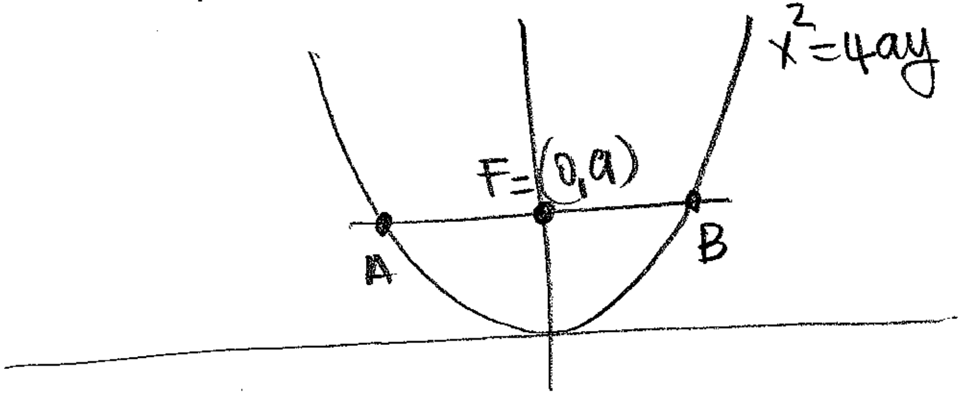
$$y^2+8y+16 = x^2-4x+y^2-4y+8$$

$$\cancel{y^2}+8y-x^2+4x-\cancel{y^2}+4y+8 = 0$$

$$-x^2+4x+12y+8 = 0$$

$$x^2-4x-12y-8 = 0$$

Ex The Line segment AB through the focus is called a focal chord Find the coordinates of A ; B



$y=a \Rightarrow x^2=4aa \Rightarrow x^2=4a^2 \Rightarrow x=\pm 2a$

thus $A=(-2a,a)$, $B(2a,a)$.

Ex Given $y^2=8x$, Find the focus point

$y^2=4(2)x$, $F=(2,0)$.