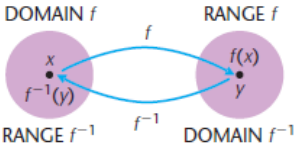
5-6 Inverse Trigonometric Functions

FACTS ABOUT INVERSE FUNCTIONS

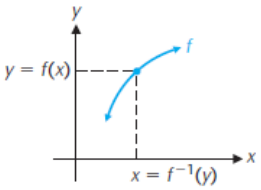
For a one-to-one function f and its inverse f^{-1}:

1. If (a, b) is an element of f, then (b, a) is an element of f^{-1}, and conversely.

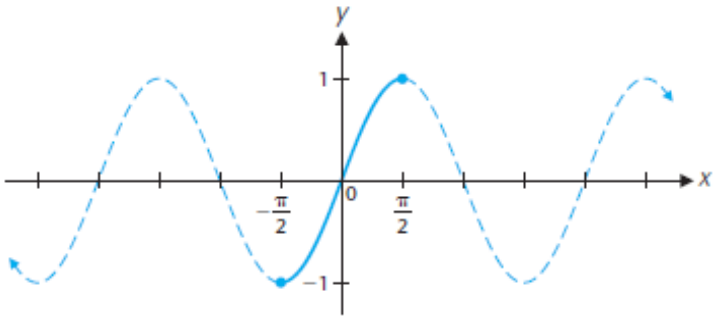
2. Range of f = Domain of f^{-1}
   Domain of f = Range of f^{-1}

3. 

4. If $x = f^{-1}(y)$, then $y = f(x)$ for y in the domain of f^{-1} and x in the domain of f, and conversely.



5. $f(f^{-1}(y)) = y$ for y in the domain of f^{-1}
   $f^{-1}(f(x)) = x$ for x in the domain of f



$y = \sin x$ is one-to-one over $[-\pi/2, \pi/2]$.

› DEFINITION 1 Inverse Sine Function

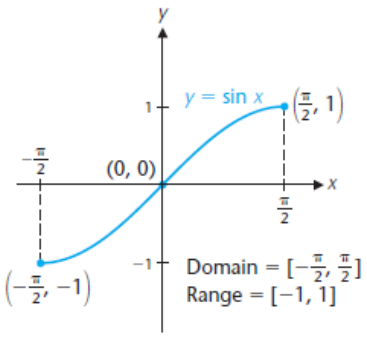
The **inverse sine function**, denoted by $\sin^{-1}$ or arcsin, is defined as the inverse of the restricted sine function $y = \sin x$, $-\pi/2 \le x \le \pi/2$. Thus,

$$y = \sin^{-1} x \text{ and } y = \arcsin x$$

are equivalent to

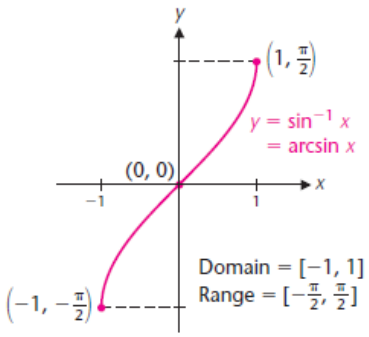
$$\sin y = x \quad \text{where} \quad -\pi/2 \le y \le \pi/2, -1 \le x \le 1$$

In words, the inverse sine of x, or the arcsine of x, is the number or angle y, $-\pi/2 \le y \le \pi/2$, whose sine is x.



Restricted sine function

(a)



Inverse sine function

(b)

SINE–INVERSE SINE IDENTITIES

$$\sin(\sin^{-1} x) = x \qquad -1 \le x \le 1 \qquad f(f^{-1}(x)) = x$$
$$\sin^{-1}(\sin x) = x \qquad -\pi/2 \le x \le \pi/2 \qquad f^{-1}(f(x)) = x$$

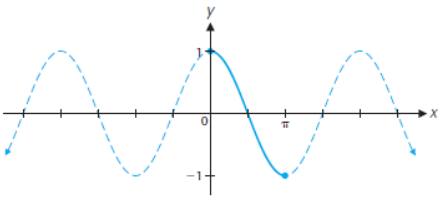
$\sin(\sin^{-1} 0.7) = 0.7$ $\qquad$ $\sin(\sin^{-1} 1.3) \neq 1.3$

$\sin^{-1}[\sin(-1.2)] = -1.2$ $\qquad$ $\sin^{-1}[\sin(-2)] \neq -2$

Find exact values without using a calculator.

(A) $\arcsin(-\frac{1}{2})$ $\qquad$ (B) $\sin^{-1}(\sin 1.2)$ $\qquad$ (C) $\cos[\sin^{-1}(\frac{2}{3})]$

Inverse Cosine Function



› DEFINITION 2 Inverse Cosine Function

The **inverse cosine function**, denoted by $\cos^{-1}$ or arccos, is defined as the inverse of the restricted cosine function $y = \cos x$, $0 \le x \le \pi$. Thus,

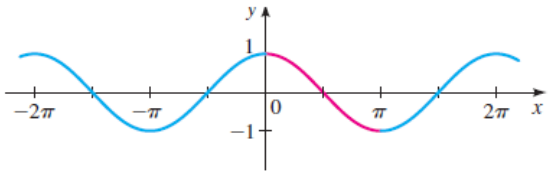
$$y = \cos^{-1} x \qquad \text{and} \qquad y = \arccos x$$

are equivalent to

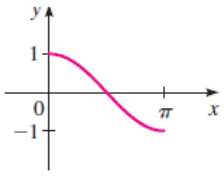
$$\cos y = x \qquad \text{where} \qquad 0 \le y \le \pi, -1 \le x \le 1$$

In words, the inverse cosine of x, or the arccosine of x, is the number or angle y, $0 \le y \le \pi$, whose cosine is x.

The Inverse Cosine Function



$y = \cos x$



$y = \cos x,\ 0 \le x \le \pi$

Definition of the Inverse Cosine Function

The **inverse cosine function** is the function $\cos^{-1}$ with domain $[-1, 1]$ and range $[0, \pi]$ defined by

$$\cos^{-1} x = y \quad \Leftrightarrow \quad \cos y = x$$

The inverse cosine function is also called **arccosine**, denoted by **arccos**.

$$\cos(\cos^{-1} x) = x \qquad \text{for } -1 \le x \le 1$$
$$\cos^{-1}(\cos x) = x \qquad \text{for } 0 \le x \le \pi$$

Evaluating the Inverse Cosine Function

Find: (a) $\cos^{-1}(\sqrt{3}/2)$, (b) $\cos^{-1} 0$, and (c) $\cos^{-1} \frac{5}{7}$.

Composing Trigonometric Functions and Their Inverses

Write $\sin(\cos^{-1}x)$ and $\tan(\cos^{-1}x)$ as algebraic expressions in x for $-1 \le x \le 1$.

Solution 1 Let $u = \cos^{-1}x$. We need to find $\sin u$ and $\tan u$ in terms of x. As in Example 3 the idea here is to write sine and tangent in terms of cosine. We have

$$\sin u = \pm\sqrt{1 - \cos^2 u} \qquad \text{and} \qquad \tan u = \frac{\sin u}{\cos u} = \frac{\pm\sqrt{1 - \cos^2 u}}{\cos u}$$

To choose the proper signs, note that u lies in the interval $[0, \pi]$ because $u = \cos^{-1}x$. Since $\sin u$ is positive on this interval, the $+$ sign is the correct choice. Substituting $u = \cos^{-1}x$ in the displayed equations and using the relation $\cos(\cos^{-1}x) = x$ gives

$$\sin(\cos^{-1}x) = \sqrt{1 - x^2} \qquad \text{and} \qquad \tan(\cos^{-1}x) = \frac{\sqrt{1 - x^2}}{x}$$

Solution 2 Let $\theta = \cos^{-1}x$, so $\cos\theta = x$. In Figure 7 we draw a right triangle with an acute angle θ, adjacent side x, and hypotenuse 1. By the Pythagorean Theorem, the remaining leg is $\sqrt{1 - x^2}$. From the figure,

$$\sin(\cos^{-1}x) = \sin\theta = \sqrt{1 - x^2} \qquad \text{and} \qquad \tan(\cos^{-1}x) = \tan\theta = \frac{\sqrt{1 - x^2}}{x}$$

Definition of the Inverse Tangent Function

The **inverse tangent function** is the function $\tan^{-1}$ with domain $\mathbb{R}$ and range $(-\pi/2, \pi/2)$ defined by

$$\tan^{-1} x = y \quad \Leftrightarrow \quad \tan y = x$$

The inverse tangent function is also called **arctangent**, denoted by **arctan**.

Thus, $\tan^{-1} x$ *is the number in the interval* $(-\pi/2, \pi/2)$ *whose tangent is* x. The following relations follow from the inverse function properties.

$$\begin{aligned} \tan(\tan^{-1} x) &= x \qquad \text{for } x \in \mathbb{R} \\ \tan^{-1}(\tan x) &= x \qquad \text{for } -\frac{\pi}{2} < x < \frac{\pi}{2} \end{aligned}$$

Find: (a) $\tan^{-1} 1$, (b) $\tan^{-1}\sqrt{3}$, and (c) $\tan^{-1}(-20)$.

Solution

(a) The number in the interval $(-\pi/2, \pi/2)$ with tangent 1 is $\pi/4$. Thus, $\tan^{-1} 1 = \pi/4$.

(b) The number in the interval $(-\pi/2, \pi/2)$ with tangent $\sqrt{3}$ is $\pi/3$. Thus, $\tan^{-1}\sqrt{3} = \pi/3$.

(c) We use a calculator to find that $\tan^{-1}(-20) \approx -1.52084$.