#3

Solutions to worksheet on Prod-sum and sum-prod Identities

Show that

$$\sin\theta\left(\sin 4\theta+\sin 6\theta\right)=\cos\theta\left(\cos 4\theta-\cos 6\theta\right).$$

LHS =

$$\sin\theta\left(\sin 4\theta+\sin 6\theta\right)=\sin\theta\left(\sin(5\theta+\theta)+\sin(5\theta-\theta)\right)$$

$$=(\sin\theta)2\cdot\frac{1}{2}\left(\sin(5\theta+\theta)+\sin(5\theta-\theta)\right)$$

$$=2\sin\theta\ \sin 5\theta\cos\theta$$

$$=2\cos\theta\left(\sin\theta\sin 5\theta\right)$$

$$=2\cos\theta\left(\frac{1}{2}\left[\cos(-4\theta)-\cos 6\theta\right]\right)$$

$$=\cos\theta\left(\cos 4\theta-\cos 6\theta\right)=\text{RHS}$$

Thus,

$$\boxed{\sin\theta\left(\sin 4\theta+\sin 6\theta\right)=\cos\theta\left(\cos 4\theta-\cos 6\theta\right)}$$

#6

$$\frac{\cos 9\theta - \cos 3\theta}{2\sin 6\theta} = \frac{-\cancel{2}\,\sin \cancel{6\theta}\,\sin 3\theta}{-2\sin\cancel{6\theta}}$$

$$= \boxed{\sin 3\theta}$$

#7

$$\frac{\cos\alpha + \cos\beta}{\sin\alpha - \sin\beta} = \frac{\cancel{2}\cos\cancel{\left(\frac{\alpha+\beta}{2}\right)}\cos\left(\frac{\alpha-\beta}{2}\right)}{\cancel{2}\sin\left(\frac{\alpha-\beta}{2}\right)\cancel{\cos\left(\frac{\alpha+\beta}{2}\right)}}$$

$$= \frac{\cos\left(\frac{\alpha-\beta}{2}\right)}{\sin\left(\frac{\alpha-\beta}{2}\right)}$$

$$= \boxed{\cot\left(\frac{\alpha-\beta}{2}\right)}$$

#8

$$1 - \cos 2\theta + \cos 6\theta - \cos 8\theta$$

$$= (\cos 0 - \cos 2\theta) + (\cos 6\theta - \cos 8\theta)$$

$$= 2\sin\theta\sin\theta + 2\sin 7\theta\sin\theta$$

$$= 2\sin\theta(\sin\theta + \sin 7\theta)$$

$$= 2\sin\theta(2\sin 4\theta\cos 3\theta)$$

$$= \boxed{4\sin\theta\cos 3\theta\sin 4\theta}$$

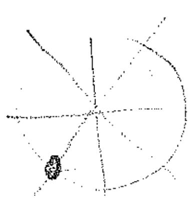
6.5 Trig Equations

Example Determine if $\theta = \frac{\pi}{4}$ is a solution of the equation $2\sin\theta + \sqrt{2} = 0$. Is $\theta = \frac{5\pi}{4}$ a solution?

$$
\begin{aligned}
2\sin\frac{\pi}{4} + \sqrt{2} &= 2\left(\frac{\sqrt{2}}{2}\right) + \sqrt{2} \\
&= \sqrt{2} + \sqrt{2} \\
&= 2\sqrt{2} \\
&\neq 0.
\end{aligned}
$$

Thus the answer is no for $\theta = \frac{\pi}{4}$

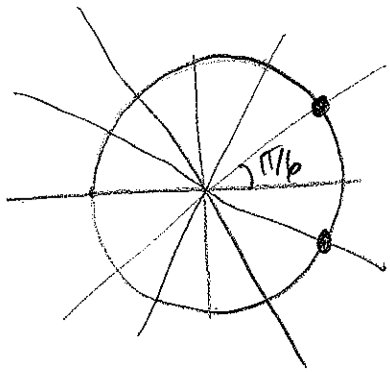
$$
\begin{aligned}
2\sin\frac{5\pi}{4} + \sqrt{2} &= 2\left(-\frac{\sqrt{2}}{2}\right) + \sqrt{2} \\
&= -\sqrt{2} + \sqrt{2} \\
&= 0
\end{aligned}
$$



Thus the answer is yes for $\theta = \frac{5\pi}{4}$

Example Solve the equation $2\cos\theta - \sqrt{3} = 0$

$$2\cos\theta = \sqrt{3} \Rightarrow \cos\theta = \frac{\sqrt{3}}{2}.$$



For example on $[0, 2\pi)$ we have 2 solutions

$$\theta = \frac{\pi}{6} \text{ and } \theta = \frac{11\pi}{6}$$

A general formula would include all coterminal angles,

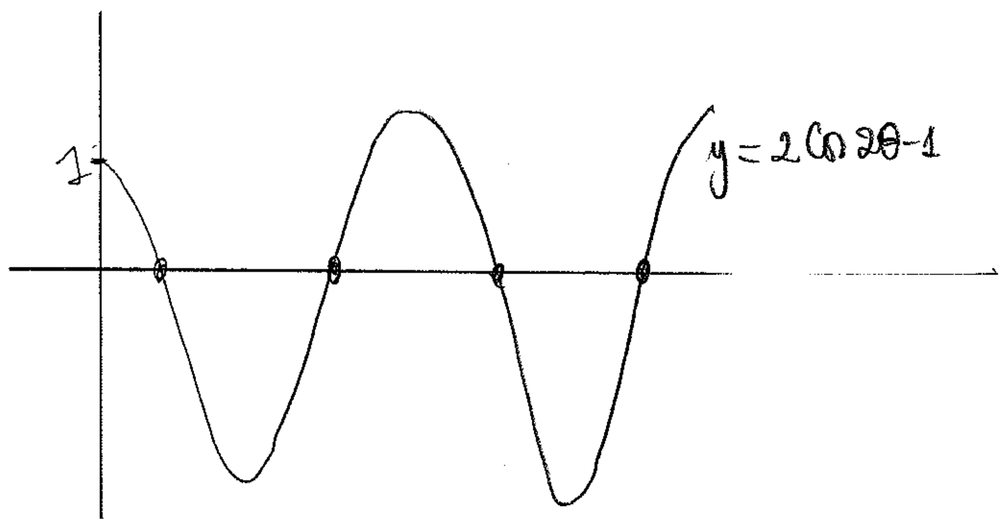
$$\theta = \frac{\pi}{6} + 2k\pi \text{ or } \theta = \frac{11\pi}{6} + 2k\pi, \quad k \in \mathbb{Z}$$

The solution set is

$$\left\{ \frac{\pi}{6} + 2k\pi, \ \frac{11\pi}{6} + 2k\pi \ \middle|\ k \in \mathbb{Z} \right\}$$

Example Solve $2\cos 2\theta - 1 = 0$, $0 \le \theta < 2\pi$

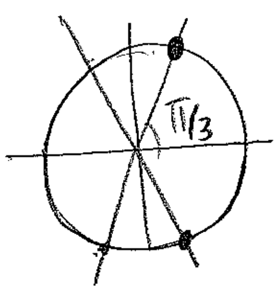
First plot the graph of $y = 2\cos 2\theta - 1$ on $[0, 2\pi)$



We expect 4 solutions.

Now, we solve

$$2\cos 2\theta = 1 \Rightarrow \cos 2\theta = \frac{1}{2}$$



$$2\theta = \frac{\pi}{3} + 2k\pi \quad \text{or} \quad 2\theta = \frac{5\pi}{3} + 2k\pi$$

$$\theta = \frac{\pi}{6} + k\pi \quad \text{or} \quad \theta = \frac{5\pi}{6} + k\pi$$

If $k = 0$, $\theta = \frac{\pi}{6}$ or $\theta = \frac{5\pi}{6}$

$k = 1$, $\theta = \frac{\pi}{6} + \pi$ or $\theta = \frac{5\pi}{6} + \pi$

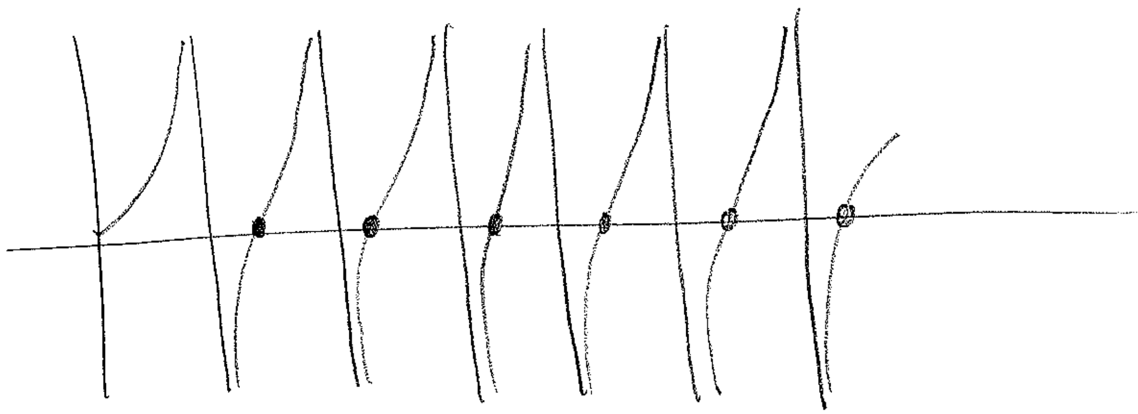
$= \frac{7\pi}{6}$ or $\theta = \frac{11\pi}{6}$

The solution set is $\left\{ \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6} \right\}$

Example Solve the equation $\sqrt{3}\tan(3\theta)+1=0$, $\theta\in[0,2\pi]$

First, in order to know how many solutions we expect, we plot

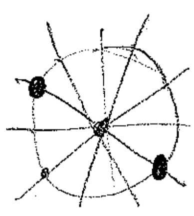
$$y=\sqrt{3}\tan(3\theta)+1 \quad \text{on} \quad [0,2\pi]$$



We expect to have 6 distinct solutions.

Now, $\quad \tan 3\theta = -\dfrac{1}{\sqrt{3}} = -\dfrac{\sqrt{3}}{3}$,

$$3\theta = \frac{5\pi}{6} + 2k\pi \quad \text{or} \quad 3\theta = \frac{11\pi}{6} + 2k\pi$$



$$\theta = \frac{5\pi}{18} + \frac{2k\pi}{3} \quad \text{or} \quad \theta = \frac{11\pi}{18} + \frac{2k\pi}{3}$$

$\frac{11}{24}$

If $k=0$, $\theta = \dfrac{5\pi}{18}$ or $\dfrac{11\pi}{18}$

If $k=1$, $\theta = \dfrac{5\pi}{18}+\dfrac{2\pi}{3}=\dfrac{17\pi}{18}$ or $\dfrac{11\pi}{18}+\dfrac{2\pi}{3}=\dfrac{23\pi}{18}$

If $k=2$, $\theta = \dfrac{5\pi}{18}+\dfrac{4\pi}{3}=\dfrac{29\pi}{18}$ or $\dfrac{11\pi}{18}+\dfrac{4\pi}{3}=\dfrac{35\pi}{18}$

$$\left\{\frac{5\pi}{18}, \frac{11\pi}{18}, \frac{17\pi}{18}, \frac{23\pi}{18}, \frac{29\pi}{18}, \frac{35\pi}{18}\right\}$$

Example Solve the equation

$$2\cos^2\theta - \cos\theta - 1 = 0, \quad \theta \in [0, 2\pi)$$

Put $x = \cos\theta$

$$2x^2 - x - 1 = 0$$

Factors of -2 adding up to -1, -2 and 1

$$2x^2 - 2x + x - 1 = 0$$

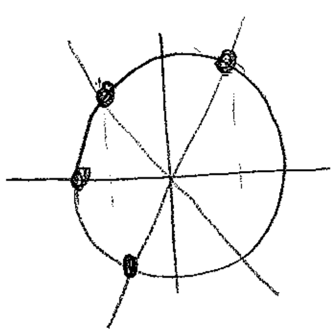
$$2x(x-1) + (x-1) = 0$$

$$(x-1)(2x+1) = 0$$

$$x - 1 = 0 \text{ or } 2x + 1 = 0$$

$$x - 1 = 0 \Rightarrow \cos\theta = 1 \Rightarrow \theta = 0$$

$$2x + 1 = 0 \Rightarrow \cos\theta = -\frac{1}{2} \Rightarrow \theta = \frac{2\pi}{3} \text{ or } \theta = \frac{4\pi}{3}.$$



Thus the solution set is

$$\boxed{\left\{0, \frac{2\pi}{3}, \frac{4\pi}{3}\right\}}$$

Worksheet on 6.5 trig equations

Name________________________________

Solve the equation on the interval $0 \le \theta < 2\pi$.

1) $2 \cos \theta + 3 = 2$

Solve the equation. Give a general formula for all the solutions.

2) $\sin \theta = 1$

3) $\tan \theta = -1$

Solve the equation on the interval $0 \le \theta < 2\pi$.

4) $4 \sin^2 \theta = 1$

5) $\sin(4\theta) = \dfrac{\sqrt{3}}{2}$

6) $2 \cos(2\theta) = \sqrt{3}$

7) $6 \csc \theta - 2 = 4$

Solve the equation. Give a general formula for all the solutions.

8) $\cos(2\theta) = \dfrac{\sqrt{2}}{2}$

Solve the equation on the interval $0 \le \theta < 2\pi$.

9) $\cos^2 \theta + 2 \cos \theta + 1 = 0$

10) $\sin^2 \theta + \sin \theta = 0$

11) $2 \sin^2 \theta - 3 \sin \theta - 2 = 0$

Answer Key
Testname: UNTITLED3

1) $\left\{\frac{2\pi}{3}, \frac{4\pi}{3}\right\}$

2) $\left\{\theta \mid \theta = \frac{\pi}{2} + 2k\pi\right\}$

3) $\left\{\theta \mid \theta = \frac{3\pi}{4} + k\pi\right\}$

4) $\left\{\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}\right\}$

5) $\left\{\frac{\pi}{12}, \frac{\pi}{6}, \frac{2\pi}{3}, \frac{7\pi}{12}, \frac{7\pi}{6}, \frac{13\pi}{12}, \frac{5\pi}{3}, \frac{19\pi}{12}\right\}$

6) $\left\{\frac{\pi}{12}, \frac{11\pi}{12}, \frac{13\pi}{12}, \frac{23\pi}{12}\right\}$

7) $\left\{\frac{\pi}{2}\right\}$

8) $\left\{\theta \mid \theta = \frac{\pi}{8} + k\pi,\ \theta = \frac{7\pi}{8} + k\pi\right\}$

9) $\{\pi\}$

10) $\left\{0, \pi, \frac{3\pi}{2}\right\}$

11) $\left\{\frac{7\pi}{6}, \frac{11\pi}{6}\right\}$