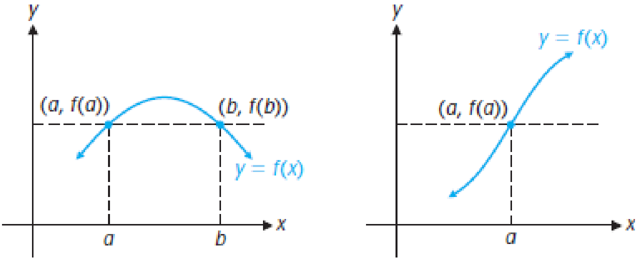
One-to-one functions

THEOREM 2 Horizontal Line Test

A function is one-to-one if and only if every horizontal line intersects the graph of the function in at most one point.

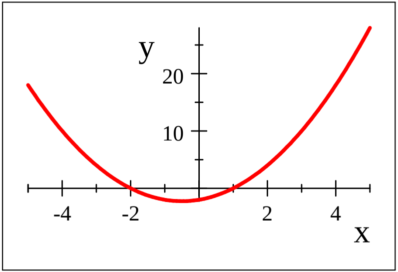


$f(a) = f(b)$ for $a \neq b$
f is not one-to-one
(a)

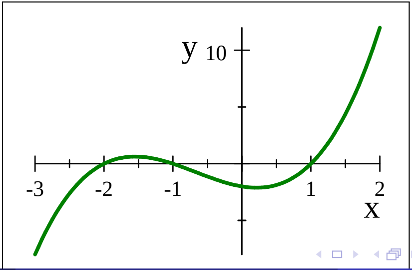
Only one point has second component $f(a)$; f is one-to-one
(b)

Example: Using the horizontal line test, determine if each function is one-to-one

1. $f(x) = (x-1)(x+2)$



2. $g(x) = (x-1)(x+2)(x+1)$



THEOREM 3 **Increasing and Decreasing Functions**

If a function f is increasing throughout its domain or decreasing throughout its domain, then f is a one-to-one function.

DEFINITION 2 **Inverse of a Function**

If f is a one-to-one function, then the **inverse** of f, denoted f^{-1}, is the function formed by reversing all the ordered pairs in f. That is,

$$f^{-1} = \{(y, x) \mid (x, y) \text{ is in } f\}$$

If f is not one-to-one, then f ***does not have an inverse function*** and f^{-1} ***does not exist.***

THEOREM 4 Properties of Inverse Functions

For a given function f, if f^{-1} exists, then

1. f^{-1} is a one-to-one function.
2. The domain of f^{-1} is the range of f.
3. The range of f^{-1} is the domain of f.

THEOREM 5 Inverse Functions and Composition

If f^{-1} exists, then

1. $f(f^{-1}(x)) = x$ for all x in the domain of f^{-1}.
2. $f^{-1}(f(x)) = x$ for all x in the domain of f.

If f and g are one-to-one functions satisfying

$$f(g(x)) = x \text{ for all } x \text{ in the domain of } g \text{ and}$$
$$g(f(x)) = x \text{ for all } x \text{ in the domain of } f$$

then f and g are inverses of one another.

Example:Decide if these two functions are inverses

$$f(x) = 3x - 7 \text{ and } g(x) = \frac{x+7}{3}$$

Finding the Inverse of a Function f

Step 1. Find the domain of f and verify that f is one-to-one. If f is not one-to-one, then stop, because f^{-1} does not exist.

Step 2. If the function is written with function notation, like $f(x)$, replace the function symbol with the letter y. Then interchange x and y.

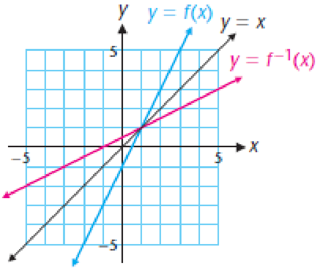
Step 3. Solve the resulting equation for y. The result is $f^{-1}(x)$.

Step 4. Find the domain of f^{-1}. Remember, the domain of f^{-1} must be the same as the range of f.

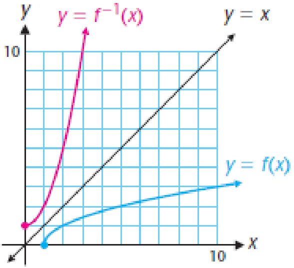
Example: Find f^{-1} for $f(x) = \sqrt{x-1}$. Make sure you check your answer

THEOREM 7 Symmetry Property for the Graphs of f and f^{-1}

The graphs of $y = f(x)$ and $y = f^{-1}(x)$ are symmetric with respect to the line $y = x$.



$f(x) = 2x - 1$
$f^{-1}(x) = \frac{1}{2}x + \frac{1}{2}$
(b)



$f(x) = \sqrt{x - 1}$
$f^{-1}(x) = x^2 + 1, x \geq 0$
(c)